STK4500: LIFE INSURANCE AND FINANCE
MANDATORY ASSIGNMENT SPRING 2023

This assignment consists of 3 exercises. The deadline is 20th of April, 2023 at 2:30 pm. To pass the assignment you need to have at least 50% correct. Good luck!

Exercise 1 (Learn your formulas). This part of the assignment have to be written by hand. Make a formulary of the most important formulas of the course so far. Please, be precise, clear and tidy.

Let us introduce the following notation:

- $\mathcal{S}$ denotes the states of the insured.
- X_t denotes the state of the insured at time t. X is assumed to be Markov.
- For states $i, j \in \mathcal{S}$, let $p_{ij}(t,s) \triangleq \mathbb{P}[X_s = j | X_t = i]$ for $s, t \geq 0$, $t \leq s$ be the transition probabilities between times and states.
- Let $A(t)$ denote a general accumulated cash flow at time $t \geq 0$.
- Let r be an instantaneous rate of return (e.g. interest rate) and $v(t) = e^{-\int_0^t r(s)ds}$ the value of one monetary unit at time t (discount factor).

Continuous time setting:

Write down:

(a) The definition of transition rates μ_{ij} between states $i, j \in \mathcal{S}$.
(b) Kolmogorov's equations to find μ_{ij} via $p_{ij}(t,s)$.
(c) Write down what is the value of the total cash flow A at $t = 0$, discounted accordingly w.r.t $v(t)$.
(d) How would you change the value of the total cash flow above if considered at any time t?
(e) Write down the retrospective and the prospective value of a general cash flow A.
(f) Define policy functions in the continuous time setting.
(g) Define the processes I_i^X and N_{ij}^X associated to X and write shortly what the describe.
(h) By means of the policy functions and the processes I_i^X and N_{ij}^X, give a formula for A that fully describes its evolution.
(i) Recast the prospective value, hereby V_t^+ of an insurance cash flow in the form given in (h).
(j) Define single premium π_0.
(k) Define a cash flow A^π of yearly continuous payments of π. Write down the associated policy function and the prospective value of such cash flow $V_t^+(A^\pi)$.
(l) Write down the formula for *expected* prospective value, given that the insured is in state i at time t, i.e. $X_t = i$.
(m) Given an insurance cash flow A, split A into the cash flow A^π like in (k) only dealing with the payment of premiums while in state $*$ and another cash

flow A^B modelling only the benefits, i.e. $A = A^\pi + A^B$. Consider now their prospective values $V_t^+(A^\pi)$ and $V_t^+(A^B)$. What is then the prospective value of A?

(n) What is the expected prospective value of the cash flow A as above?
(o) Explain what the equivalence principle is?
(p) How would you find the yearly premium π in (m)?
(q) Write down Thiele's differential equation.

Discrete time setting:

(a) Follow the same steps as above and write down the formulas for the discrete time case.

Exercise 2 (Friend Group Survival). Consider a group of $N \in \mathbb{Z}$, $N \geq 1$ friends all with the same age and individual mortality, hereby denoted as μ. Let $Z = \{Z_t\}_{t\geq 0}$ be the continuous time (regular) Markov chain which counts the number of living friends in this group, by time t. The state space of Z is then clearly $\mathcal{S} = \{0, 1, \ldots, N\}$. Define

$$p_{m\,n}(t, s) \triangleq \mathbb{P}[Z_s = n | Z_t = m]$$

and let

$$p(t, s) = e^{-\int_t^s \mu(u)du}$$

be the survival probability of an individual. Lastly we will assume that the lifespans of the friends are independent.

(a) Prove that for every $t \geq 0$ we have

$$\mu_{m\,n}(t) = 0,$$

for every $m, n \in \mathcal{S}$, $|m - n| \geq 2$ or $n = m + 1$, and that

$$\mu_{m\,m-1}(t) = m\mu(t),$$

for every $m \in \mathcal{S} \setminus \{0\}$.

Solution: On the one hand, we have $\mu_{m\,n}(t) = 0$ whenever $n \geq m + 1$ because none of the friends are capable of resurrection. On the other hand, we have $\mu_{m\,n}(t) = 0$ for $n \leq m - 2$ because our Markov process cannot *jump* twice at the same time. To see this, assume that $X^{(1)}$ and $X^{(2)}$ are the states of two friends. Then

$$\begin{aligned}
\mu_{(*,*),(\dagger,\dagger)}(t) &= \lim_{\substack{h\to 0\\ h>0}} \frac{\mathbb{P}[X^{(1)}_{t+h}=\dagger, X^{(2)}_{t+h}=\dagger | X^{(1)}_t = *, X^{(2)}_t = *]}{h} \\
&= \lim_{\substack{h\to 0\\ h>0}} \frac{\mathbb{P}[X^{(1)}_{t+h}=\dagger | X^{(1)}_t = *]\mathbb{P}[X^{(2)}_{t+h}=\dagger | X^{(2)}_t = *]}{h} \\
&= \lim_{\substack{h\to 0\\ h>0}} \frac{\mathbb{P}[X^{(1)}_{t+h}=\dagger | X^{(1)}_t = *]}{h} \lim_{\substack{h\to 0\\ h>0}} \mathbb{P}[X^{(2)}_{t+h}=\dagger | X^{(2)}_t = *] \\
&= \mu^{(1)}_{*\dagger}(t) \lim_{\substack{h\to 0\\ h>0}} \mathbb{P}[X^{(2)}_{t+h}=\dagger | X^{(2)}_t = *] \\
&= \mu^{(1)}_{*\dagger}(t)\cdot 0 \\
&= 0.
\end{aligned}$$

Iterating this argument (or conditioning on all possible pairs), we see that two or more friends cannot die instantaneously in our Markov modelling framework. Hence, $\mu_{mn}(t) = 0$ for all $n \le m-2$.

To prove that $\mu_{m\,m-1}(t) = m\mu(t)$ for $m \in \mathcal{S}\setminus\{0\}$ we make the following observation: by independence the probability that exactly *one* specific friend dies during the time interval $[t,s]$, $s \ge t$ and the rest survive is given by

$$p_{**}(t,s)^{m-1} p_{*\dagger}(t,s).$$

Thus the probability that any one of among m friends die is given by

$$\mathbb{P}[Z_s = m-1 | Z_t = m] = m p_{**}(t,s)^{m-1} p_{*\dagger}(t,s).$$

Now the transition rate is

$$\begin{aligned}
\mu_{m\,m-1}(t) &= \lim_{\substack{h\to 0\\ h>0}} \frac{\mathbb{P}[Z_{t+h} = m-1 | Z_t = m]}{h} \\
&= \lim_{\substack{h\to 0\\ h>0}} \frac{m p_{**}(t,t+h)^{m-1} p_{*\dagger}(t,t+h)}{h} \\
&= m \lim_{\substack{h\to 0\\ h>0}} p_{**}(t,t+h)^{m-1} \lim_{\substack{h\to 0\\ h>0}} \frac{p_{*\dagger}(t,t+h)}{h} \\
&= m\mu(t).
\end{aligned}$$

(b) Argue that for every $t, s \ge 0$, $s \ge t$,

$$p_{m\,n}(t,s) = 0,$$

for every $m, n \in S$, $n \ge m+1$ and that,

$$p_{m\,n}(t,s) = \binom{m}{n} p(t,s)^n (1-p(t,s))^{m-n},$$

for every $m, n \in S$, $n \leq m$ and show that $s \mapsto p_{mn}(t,s)$ satisfies Kolmogorov's forward equation.

Solution: We first note that since none of the friends are capable of resurrection we must have that the number of living friends at time s are exactly the number of friends who were alive at t and survived. Since the survival probability of an individual in the period $[t,s]$ is $p(t,s)$ we see by independence of lifespans that the events

$$B_i = \{\text{friend } i \text{ survived from } t \text{ to } s\},$$

are i.i.d. Bernoulli trials. The number of living friends at time s, given that $Z_t = m$, is then binomially distributed with probability mass function

$$p_{m\,n}(t,s) = \binom{m}{n} p(t,s)^n (1-p(t,s))^{m-n}. \tag{1}$$

Komogorov's forward equation for $p_{m\,n}(t,s)$ is given by

$$\frac{d}{ds} p_{m\,n}(t,s) = -p_{m\,n}(t,s)\mu_n(s) + \sum_{\substack{k\in S\\ k\neq n}} p_{m\,k}(t,s)\mu_{k\,n}(s), \quad p_{m\,n}(t,t) = 0, \quad m \neq n.$$

Observe that, as we proved in item (a), we have $p_{m\,k}(t,s) = 0$ for all $k \geq m+1$ and $\mu_{k\,n}(s) = 0$ for all $k \geq n+2$. So Kolmogorov's equation simplifies to

$$\frac{d}{ds} p_{m\,n}(t,s) = -p_{m\,n}(t,s)\mu_n(s) + p_{m\,n+1}(t,s)\mu_{n+1\,n}(s), \quad p_{m\,n}(t,t) = 0, \quad m \neq n.$$

It is readily checked from (1) that $p_{m\,n}(t,t) = 1$ if $m \neq n$. On the one hand we have

$$\begin{aligned}
\frac{d}{ds} p_{m\,n}(t,s) &= \binom{m}{n} n p(t,s)^{n-1} \frac{d}{ds} p(t,s)(1-p(t,s))^{m-n} \\
&\quad - \binom{m}{n} p(t,s)^n (m-n)(1-p(t,s))^{m-n-1} \frac{d}{ds} p(t,s) \\
&= -\binom{m}{n} n\mu(s) p(t,s)^n (1-p(t,s))^{m-n} \\
&\quad + \binom{m}{n} \mu(s) p(t,s)^{n+1} (m-n)(1-p(t,s))^{m-n-1},
\end{aligned}$$

where we used that $\frac{d}{ds} p(t,s) = -\mu(s)p(t,s)$. Now, observe that $n\mu(s) = \mu_{n\,n-1}(s)$. On the other hand, $\mu_{n\,n-1}(s) + \mu_{n\,n}(s) = 0$ so $\mu_{n\,n-1}(s) = -\mu_{n\,n}(s)$ and by convention $-\mu_{n\,n}(s) = \mu_n(s)$. Also, observe that

$$\begin{aligned}
&\binom{m}{n} \mu(s) p(t,s)^{n+1} (m-n)(1-p(t,s))^{m-n-1} \\
&\quad = (n+1)\mu(s) \binom{m}{n+1} p(t,s)^{n+1} (1-p(t,s))^{m-(n+1)} \\
&\quad = p_{m\,n+1}(t,s)\mu_{n+1\,n}(s),
\end{aligned}$$

and the proof of the case $m \neq n$ follows.

For $m = n$ Kolmogorov's equation reduces to

$$\frac{d}{ds}p_{m\,m}(t,s) = -p_{m\,m}(t,s)\mu_m(s) = -p_{m\,m}(t,s)m\mu(s), \quad p_{m\,m}(t,t) = 1,$$

Which we have seen to have the solution

$$p_{m\,m}(t,s) = e^{-\int_t^s m\mu(u)du} = p(t,s)^m = \binom{m}{m} p(t,s)^m(1-p(t,s))^{m-m}.$$

Note that proving that (1) satisfies Kolmogorov's equation is also a rigorous proof of (1) being the transition probabilities of the Markov process Z.

Exercise 3 (A Tontine of Friends in Continuous Time). A tontine is an old investment strategy named after Neapolitan banker Lorenzo de Tonti, who is popularly credited with inventing it in France in 1653.

The main idea of a tontine is that a group of people all pay a single lump sum and invest it into a shared fund. Every year the net profit of the fund is distributed amongst the investors as yearly dividends. Whenever any of the initial investors die, the yearly dividends will then be distributed between fewer and fewer people, thus increasing the payout for those who remain alive.

In this exercise we will assume that an insurance company oversees and manages a tontine insurance scheme with N friends under the same conditions and notations as in the previous exercise. We will focus on one of these participants, referred to as the *chosen one*. As such we model the state of everyone involved by the Markov process X with the state space

$$\mathcal{S} = \{0, 1, \dots, N-1\} \times \{*, \dagger\}.$$

Here the state $(m, *)$ means that m participants, other than the chosen one, are alive and that that the chosen one is still living, while $(m, \dagger)$ means that m participants are alive, but our chosen one is dead.

(a) **Transition Probabilities.**

Argue in an almost analogous way as in Exercise 1, that

$$\begin{aligned} \mu_{(m,*)\,(m-1,\dagger)}(t) &= 0, \\ \mu_{(m,*)\,(m,\dagger)}(t) &= \mu(t), \\ \mu_{(m,*)\,(m-1,*)}(t) &= \mu_{(m,\dagger)\,(m-1,\dagger)}(t) = m\mu(t), \end{aligned}$$

for every $t \geq 0$ and $m \in \{0, 1, \dots, N-1\}$. As a result, argue that for every $t, s \geq 0$, $s \geq t$,

$$p_{(m,j)\,(n,j)}(t,s) = 0,$$

for every $m, n \in \mathcal{S}$, $n \geq m+1$, $j \in \{*, \dagger\}$ and that,

$$p_{(m,*)\,(n,*)}(t,s) = \binom{m}{n} p(t,s)^{n+1}(1-p(t,s))^{m-n},$$

and

$$p_{(m,*)\,(n,\dagger)}(t,s) = \binom{m}{n} p(t,s)^n (1-p(t,s))^{m-n+1},$$

for every $m, n \in S$, $n \leq m$.

Solution: Note that by independent lives we have

$$p_{(m,*)\,(n,*)}(t,s) = p_{m\,n}(t,s)p(t,s),$$

which, along with the same arguments as Exercise 1 (a), yields the desired results.

We assume that the contract starts at time $t = 0$ and that all participants pay a single premium π_0 at this time. The premiums are invested into a fund managed by the insurance company. Let $S = \{S(t), t \geq 0\}$ denote the value of a fund. We assume that the value of the fund evolves according to

$$S(t) = S(0)e^{\rho t}, \quad t \geq 0,$$

for some $\rho \in \mathbb{R}$, $\rho > 0$.

At retirement time $T_0 \geq 0$, we start paying out the returns from the fund to the living participants. Observe that the return on $[t, t+dt]$ is given by $S(t+dt) - S(t) = S(t)\left[e^{\rho dt} - 1\right]$ and if dt is infinitesimally small we have $e^{\rho dt} - 1 = \rho dt + O(dt^2)$. Hence, the instantaneous return at time t is given by $\rho S(t)$. In particular, our participants receive instantaneously $\rho S(T_0)$ after $t \geq T_0$ to be distributed among all surviving participants. We can also note that since all the profits of the fund are distributed the value of the fund stagnates at time T_0 and we have for all $t \geq T_0$ that

$$S(t) = S(T_0) = S(0)e^{\rho T_0}.$$

From now on, we assume a constant interest rate $r > 0$ that the insurance company uses to price its policies.

(b) **Policy Functions.** Take the perspective of the chosen one and therefore ignore all payments that are not going to them. Write down the policy functions for the chosen one's contract without taking into account premiums, yet. Note that since we are taking the perspective of the chosen one we have $a_{(m,\dagger)}(t) = 0$ for all m.

Solution: The instantaneous return generated by the fund is $\rho S(T_0)$ as explained in the exercise. In state $(m,*)$ we have exactly $m+1$ survivors to redistribute the wealth among. Hence, each obtain $\frac{\rho S(T_0)}{m+1}$ at time $t \geq T_0$. The accumulated payments in state $(m,*)$ are therefore given by

$$a_{(m,*)}(t) = \begin{cases} 0 & \text{for } t < T_0 \\ \frac{\rho S(T_0)}{m+1}(t - T_0) & \text{for } t \geq T_0 \end{cases}$$

(c) **Cost of the insurance.** Show that the cost of this insurance, i.e., the present value at each time of the future liabilities is given by

$$V^+_{(m,*)}(t) = \rho S(0)e^{\rho T_0} \sum_{n=0}^{m} \int_{t \vee T_0}^{\infty} \binom{m}{n} \frac{p(t,s)^{n+1}(1-p(t,s))^{m-n}}{n+1} e^{-r(s-t)} ds$$

Solution: The present value of the cash flow generated by the policy functions $a_{(m,*)}$ is given by

$$V^+_{(m,*)}(t) = \frac{1}{v(t)} \sum_{n=0}^{m} \int_t^\infty v(s) p_{(m,*)\,(n,*)}(t,s) da_{(n,*)}(s),$$

where $v(t) = e^{-rt}$ is the discount factor. Noting that the policy functions are a.e. differentiable and that $da_{(n,*)}(s) = \mathbb{1}_{s \geq T_0} \frac{\rho S(T_0)}{n+1} ds$ and using the expression for $p_{(m,*)\,(n,*)}(t,s)$ from item (a), we obtain the desired expression.

(d) **A simpler formula.** Show that the present value $V^+_{(m,*)}(t)$ can be written in the following simplified form,

$$V^+_{m,*}(t) = \frac{\rho S(0) e^{\rho T_0}}{m+1} \int_{t \vee T_0}^\infty \left(1 - (1 - p(t,s))^{m+1}\right) e^{-r(s-t)} ds$$

Note that for numerical implementations this form might be easier and faster to use.

Solution: We note that

$$\binom{m}{n} = \binom{m+1}{n+1} \frac{n+1}{m+1},$$

and that by total probability we have

$$\sum_{k=0}^{m+1} \frac{p_{m+1\,k}(t,s)}{m+1} = 1.$$

where $p_{m+1\,k}$ is the transition probability in Exercise 1 (b). This means that, if we substitute $k = n+1$, we have

$$\begin{aligned}
&\sum_{n=0}^{m} \binom{m}{n} \frac{p(t,s)^{n+1}(1-p(t,s))^{m-n}}{n+1} \\
&= \sum_{n=0}^{m} \binom{m+1}{n+1} \frac{p(t,s)^{n+1}(1-p(t,s))^{m+1-(n+1)}}{m+1} \\
&= \sum_{k=1}^{m+1} \binom{m+1}{k} \frac{p(t,s)^{k}(1-p(t,s))^{m+1-k}}{m+1} \\
&= \sum_{k=1}^{m+1} \frac{p_{m+1\,k}(t,s)}{m+1} \\
&= \frac{1 - p_{m+1\,0}(t,s)}{m+1} \\
&= \frac{1 - (1-p(t,s))^{m+1}}{m+1},
\end{aligned}$$

Plugging this into the equation from item (c) yields the desired result. We can interpret this equation by considering the payouts the insurance company is liable for, regardless of how many are still alive the insurance company must pay the same total rate of $\rho S(T_0)$ as long as at least one friend is still alive. This means that the probability the insurance company has to pay is equal to the probability of at least one survivor ($= 1 - (1 - p(t, s))^{m+1}$), applying the explicit formula for reserves and dividing by $m + 1$ to get the reserves per person yields the same formula.

(e) **Single and yearly premiums.** Compute the single and yearly premiums that our chosen one has to pay to enter this policy. The yearly premiums are only paid until retirement time T_0.

Solution: The single premium is simply the initial expected cost of the policy, given that all members are alive, that is

$$\pi_0 = V^+_{(N-1,*)}(0) = \frac{\rho S(0) e^{\rho T_0}}{N} \int_{T_0}^{\infty} \left(1 - (1 - p(0, s))^N\right) e^{-rs} ds.$$

Here, $S(0)$ represents the total amount that our participants wish to buy from the fund, so $\frac{S(0)}{N}$ is how much money is invested on their behalf. Note that the initial value of the contract differs from the initial value invested into the fund.

For computing the yearly premiums, we design an artificial policy that only pays 1 monetary unit continuously on $[0, T_0]$ as long as the chosen one is alive. This corresponds to

$$a^{Prem}_{(m,*)}(t) = \begin{cases} -t, & t \in [0, T_0] \\ -T_0, & t \geq T_0 \end{cases} \qquad m \in \mathcal{S}.$$

Hence, the present value of the yearly premiums is given by

$$\begin{aligned} \pi V^+_{(m,*)}(t, A^{Prem}) &= -\pi \frac{1}{v(t)} \sum_{n=0}^{m} \int_t^{\infty} v(s) p_{(m,*)\,(n,*)}(t, s) da^{Prem}_{(n,*)}(s) \\ &= -\pi \frac{1}{v(t)} \sum_{n=0}^{m} \int_{t \vee T_0}^{\infty} v(s) p_{(m,*)\,(n,*)}(t, s) ds \\ &= -\pi \frac{1}{v(t)} \int_{t \vee T_0}^{\infty} v(s) p(t, s) ds, \end{aligned}$$

where we used that $\sum_{n=0}^{m} p_{(m,*)\,(n,*)}(t, s) = p(t, s)$.

The equivalence principle says that the initial value of the insurance should match the initial value of the future paid-in premiums (cash in should match cash in). So we must have

$$\pi V^+_{(N-1,*)}(0, A^{Prem}) + V^+_{(N-1,*)}(0) = 0.$$

Hence, the yearly premium is

$$\pi = \frac{\frac{\rho S(0) e^{\rho T_0}}{N} \int_{T_0}^{\infty} \left(1 - (1 - p(0, s))^N\right) e^{-rs} ds}{\int_{T_0}^{\infty} v(s) p(0, s) ds}.$$

The prospective reserve is given by

$$\pi V^+_{(m,*)}(t, A^{Prem}) + V^+_{(m,*)}(t).$$

(f) **Thiele's Equation.** Derive an ordinary differential equation for the present value of the policy $V^+_{(m,*)}(t)$ for $t \geq 0$.

Solution: Writing down Thiele's differential equation with transition payouts we get, for $t \geq 0$,

$$\frac{dV^+_{(m,*)}(t)}{dt} = rV^+_{(m,*)}(t) - \dot{a}_{(m,*)}(t) - \sum_{\substack{n\in\mathcal{S}\\ n\neq m}} \mu_{(m,*)\,(n,*)}(t) \left(V^+_{(n,*)}(t) - V^+_{(m,*)}(t) \right).$$

Plugging in the computed values and using the relations for $\mu_{(m,j)\,(n,j)}$, $j \in \{*, \dagger\}$ from item (a) we get

$$\frac{dV^+_{(m,*)}}{dt} = rV^+_{(m,*)} - \mathbb{1}_{T_0\leq t}\rho N\pi_0 e^{\rho T_0}/(m+1) + \mu(t)V^+_{(m,*)} - \mathbb{1}_{m>0} m\mu(t)(V^+_{(m-1,*)} - V^+_{(m,*)}).$$

(g) **A numerical example.**

From now on and until the end of the assignment, let $r = 0.03, \rho = 0.07, S(0) = 100\,000, N = 10, T_0 = 40$. Furthermore, let $\mu(t)$ be given by the K2013 mortality rates from Finanstilsynet at time $t + 2022$ for a male aged 30 at the beginning of 2022.

Using these parameters compute the single and yearly premiums as described in exercise (e). Plot the present value $V^+_{(N-1,*)}(t)$, along with the reserves for the premiums and the total reserves for $t \in [0, 100]$.

Solution: We implement the formula for the present value given in (d) and the premium reserves given in (e), producing the following plot. Code is included in the appendix.

We also get the following values

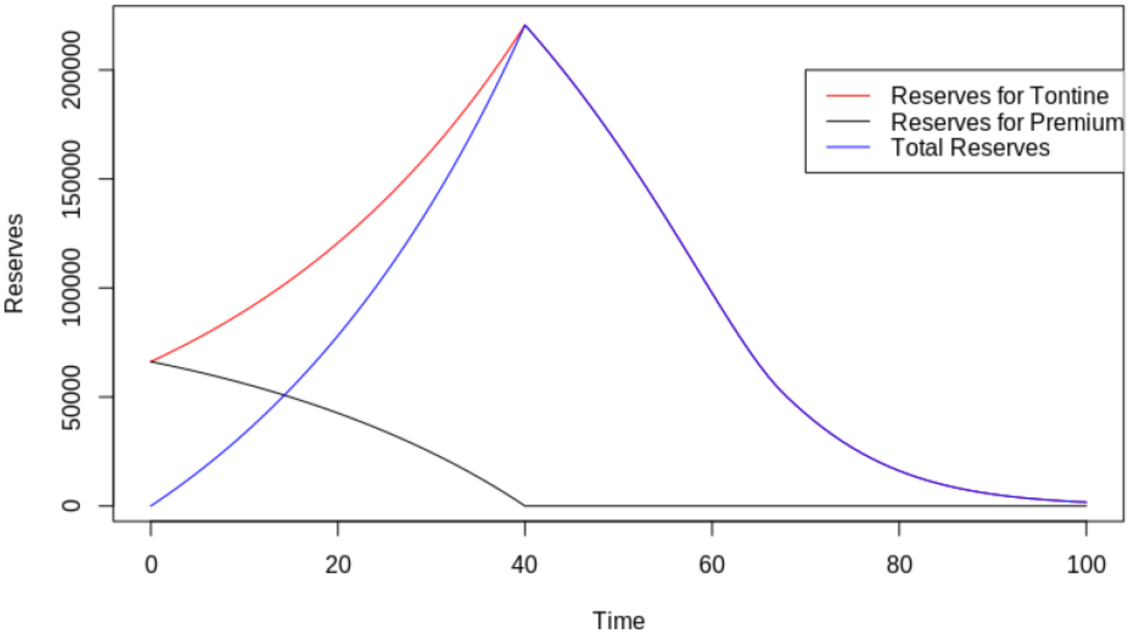
$$V^+_{(N-1,*)}(0, A^{Prem}) = -23$$
$$\pi_0 = V^+_{(N-1,*)}(0) = 66\,209$$
$$\pi = 2\,877$$

(h) **Lifespan Simulation.**

An alternative way of computing the reserves is by the simulation of lifespans.

Simulate 10 000 outcomes and compute the (discounted) costs for each of them. Plot the histogram of these values. Compute the average of these payouts. What do you obtain?

Now, simulate 20 tontine outcomes and compute the cost average, as before. Repeat this procedure 10 000 times and plot the histogram of the sample means. What distribution do you see and why?



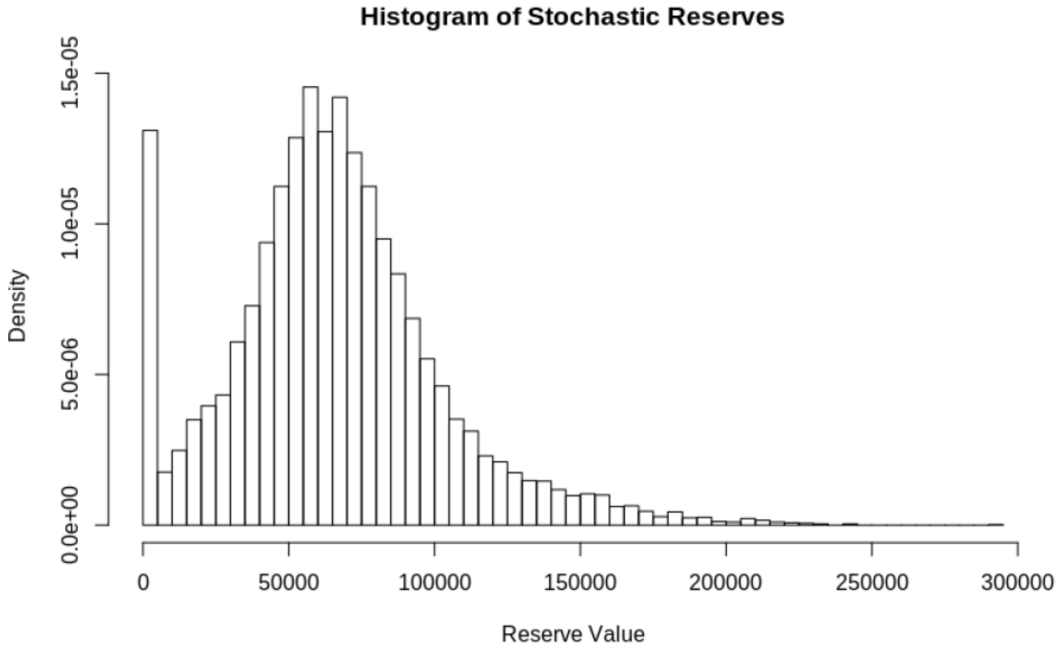
Solution: As before, code is included in the appendix.

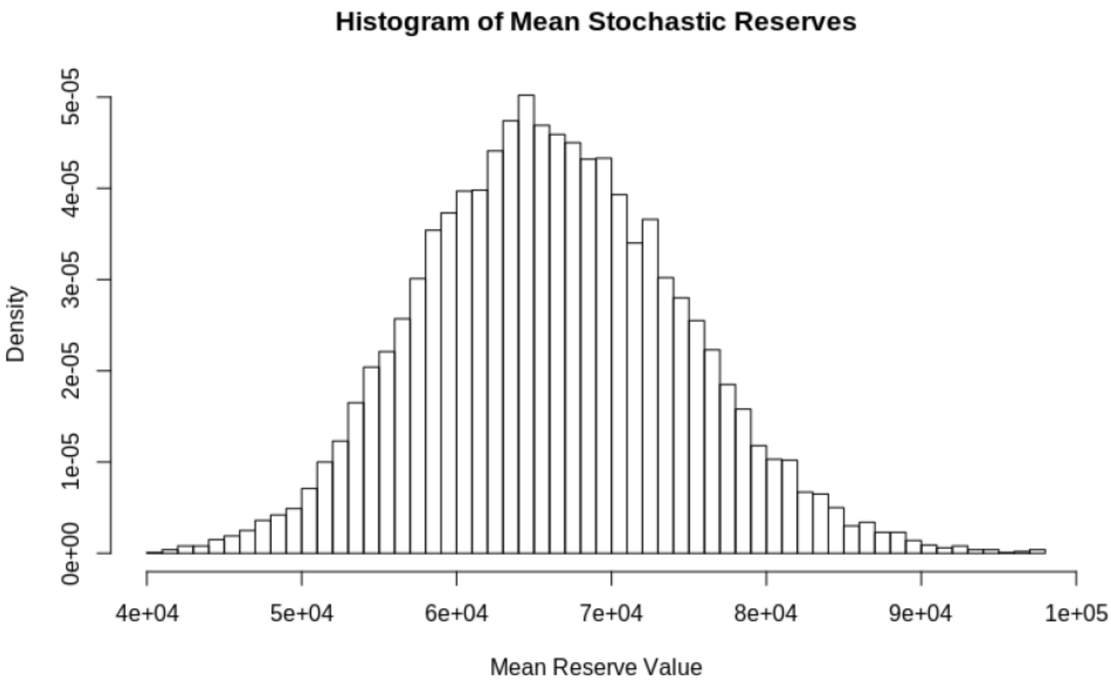
The idea behind the code is to consider the payouts between times of death. For a set of 10 death times we pick the first one to be the death time of the chosen one, denoted τ, we then sort the death times in increasing order and include 0 as a starting time, i.e. $\{\tau_i\}_{i=1}^{10}, \tau_i \leq \tau_{i+1}, \tau_0 = 0$. We then note that in the interval $[\tau_i, \tau_{i+1}]$ there are $N - i$ survivors allowing us to compute a realisation of the stochastic reserve as

$$\begin{aligned}
V_t^+ &= \sum_{i=0}^{9} \mathbb{1}_{\tau_i < \tau} \frac{\rho S(0) e^{\rho T_0}}{N - i} \int_{T_0 \vee t_i}^{T_0 \vee t_{i+1}} e^{-rs} ds \\
&= \sum_{i=0}^{9} \mathbb{1}_{\tau_i < \tau} \frac{\rho S(0) e^{\rho T_0}}{N - i} \frac{e^{-r(T_0 \vee t_i)} - e^{-r(T_0 \vee t_{i+1})}}{r} \\
&= \sum_{i=0}^{9} \frac{\rho S(0) e^{\rho T_0}}{N - i} \frac{e^{-r(\tau \wedge (T_0 \vee t_i))} - e^{-r(\tau \wedge (T_0 \vee t_{i+1}))}}{r}
\end{aligned}$$

This method allows us to simulate the distribution of the stochastic reserves as well as the cost average of twenty such contracts.

We note the skewed distribution of the reserve along with a spike at zero caused by the chance of dying before T_0. We also have an approximately normal distribution of the mean stochastic reserve as a consequence of the central limit theorem.





Appendix/Code

General Functions.

```
#################### Functions For Mortality ####################
# G: Denotes gender (0:Male,1:Female)
# x: Denotes age in year 2013, not current age.
# t, s: Denotes calendar year (assumed t, s > 2013)
```

```
w <- function(G, x){
 if(G==0){
  return( min(2.671548-0.17248*x+0.001485*x^2,0) )
 }
 if(G==1){
  return( min(1.287968-0.10109*x+0.000814*x^2,0) )
 }
}

mu.kol.2013 <- function(G, x){
 #men
 if(G==0){
  return( (0.241752+0.004536*10^(0.051*x))/1000 )
 }
 #women
 if(G==1){
  return( (0.085411+0.003114*10^(0.051*x))/1000 )
}
}

mu.kol <- function(G, x,t){
 return(mu.kol.2013(G, x+t-2013)
 *(1+w(G, x+t-2013)/100)^(t-2013))
}

# making sure infinities arent a problem
p_surv <- function(G,x,t,s){
 if (mu.kol(G=G,x=x,t=s)==Inf) {
  if (s==t) {return(1)}
  else {return(0)}
 }
 temp_int <- integrate(f=Vectorize(mu.kol),
 lower=t,upper=s,G=G,x=x)$value
 return(exp(-temp_int))
}

# transition probabilities from exercise 1
probs <- function(m,n,t,s,G=0,x=21){
 B <- p_surv(G=G,x=x,t=t+2022,s=s+2022)
 return(choose(m,n)*(B^n)*((1-B)^(m-n)) )
}
################# Functions for Reserves #################


Vcont <- function(t,m,rho=0.07,V0=1,r=0.03,min.time=40,G=0,x=21){
 VT <- V0*exp(min.time*rho)
 tempfunc <- function(s){exp(-r*(s-t)) *
```

```
  (1-(1-p_surv(0,21,t+2022,s+2022))^(m+1)) * rho*(VT/(m+1))}
 tempint <- integrate(Vectorize(tempfunc),
  lower = max(t,min.time),upper = Inf)$value
return(tempint)
}

annuity <- function(P=1,t=0,T0=40,TT=Inf,r=0.03){
 if (t>=TT){return(0)}
 tempfunc <- function(s){ exp(-r*(s-t)) * P * p_surv(0,21,t+2022,s+2022)}
 tempval <- integrate(Vectorize(tempfunc), lower = max(t,T0),
  upper = TT)$value
return(tempval)
}
```

Code for 2g.

```
Vlist <- numeric(101)
for (i in 0:100) {
   Vlist[i+1] <-
       Vcont(t=i,m=9,rho=0.07,V0=100000,r=0.03,min.time=40,G=0,x=21)
}
plot(0:100,Vlist,xlab = "Time",ylab = "Reserves",type = "l",col="red")
pi0 <- Vlist[1]

Vtilde <- annuity(P=1,t=0,T0=0,TT=40,r=0.03)
pic <- pi0/Vtilde


Vlist2 <- numeric(101)
for (i in 0:100) {
   Vlist2[i+1] <- annuity(P=pic,t=i,T0=0,TT=40,r=0.03)
}
lines(0:100,Vlist2)
lines(0:100,Vlist-Vlist2,col="blue")
legend(x=70,y=200000 ,legend = c("Reserves for Tontine ",
   "Reserves for Premiums","Total Reserves"),
col = c("red","black","blue"),lty = 1)
```