STK4500: Life Insurance and Finance

Exercise list 1

Exercise 1.1

Let S_n be the price of a stock at day $n = 0, 1, 2, \ldots$ ($n = 0$ present month). Suppose that $S_n \in \{0\$, 10\$, 15\$\}$ for all n and that the current stock price is 10\$, i.e. $S_0 = 10$ \$. Further assume that stock prices $\{S_n\}_{n \in \mathbb{N}}$ are modeled by a Markov chain with transition probability matrix P given by

$$P = \begin{pmatrix} p_{0\$\ 0\$} & p_{0\$\ 10\$} & p_{0\$\ 15\$} \\ p_{10\$\ 0\$} & p_{10\$\ 10\$} & p_{10\$\ 15\$} \\ p_{15\$\ 0\$} & p_{15\$\ 10\$} & p_{15\$\ 15\$} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0.02 & 0.68 & 0.3 \\ 0.01 & 0.64 & 0.35 \end{pmatrix}$$

Compute $P(S_5 = 15\ \$ | S_0 = 10\ \$)$. Compute the expected number of months until bankruptcy (check theory on Markov chains). Compute both quantities using R or any other programming language.

Exercise 1.2 (Compensated Poisson process)

Let $N = \{N(t), t \geqslant 0\}$ be a Poisson process with intensity $\lambda > 0$ on a probability space $(\Omega, \mathcal{F}, P)$, that is, $N = \{N(t), t \geqslant 0\}$

(i) has right-continuous sample paths with existing left limits (càdlàg paths)

(ii) starts at 0

(iii) has independent and stationary increments

(iv) $N(t)$ is Poisson distributed with parameter λt, that is

$$P(N(t) = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}, \quad n = 0, 1, \ldots$$

Show that the compensated Poisson process is a martingale with respect to the filtration $\mathcal{F}_t = \sigma(N(s), 0 \leqslant s \leqslant t)$

Exercise 1.3 (De Moivre's martingale)

Suppose a coin is unfair and denote by p the probability heads and by $q = 1 - p$ the probability of tails. Let $X_{n+1} = X_n \pm 1$ with $+$ in case of heads obtaining heads and $-$ in case of obtaining tails. Let $Y_n = (q/p)^{X_n}$, $n \geqslant 1$. Show that the process $Y = \{Y_n, n \geqslant 1\}$ is a martingale w.r.t. the filtration generated by $X_1, \ldots, X_n$, i.e. prove that Y is adapted to the filtration generated by $X_1, \ldots, X_n$, $E[|Y_n|] \leqslant \infty$ for every $n \geqslant 1$ and finally $E[Y_{n+1}|\mathcal{F}_n] = Y_n$ for every $n \geqslant 1$.

Exercise 1.4

Let X_0 be a random variable with values in the countable state space S. Let $Y_1, Y_2, \ldots$ be a sequence of i.i.d. with uniform distribution on the interval $[0, 1]$. Consider a given function

$$G : S \times [0, 1] \to S$$

and define recursively

$$X_{n+1} = G(X_n, Y_{n+1}), \quad n \geqslant 0.$$

Show that $X = \{X_n\}_{n \geqslant 0}$ is a Markov chain and express its transition matrix in terms of G. Can all (time-homogeneous) Markov chains with index set $J = \{0, 1, 2, \ldots\}$ be realized in this way? How would you simulate a Markov chain? Try a function G of your own choice and simulate X.

Exercise 1.5 (Permanent disability model)

Assume in this model that the state of the insured $X_t \in S$ is modeled by a regular Markov chain with state space $S = \{*, \diamond, \dagger\}$ where $* =$"active", $\diamond =$"disabled" and $\dagger =$"dead". The permanent disability model for an insurance policy provides some of the following benefits: (i) an annuity while permanently disabled, (ii) a lump sum on becoming permanently disabled and (iii) a lump sum on death. Premiums are paid while the insured is healthy ,i.e. active. An important feature of this model is that disablement is permanent, that is $p_{\diamond *}(s, t) = 0$, $s \leqslant t$.

(i) Suppose the transition rates for this model are all constants and given by

$$\mu_{*\diamond}(t) = 0.0279 \quad \mu_{*\dagger}(t) = 0.0229 \quad \mu_{\diamond\dagger}(t) = \mu_{*\dagger}(t).$$

Calculate $p_{**}(x, x + 10)$ and $p_{*\diamond}(x, x + 10)$ for $x = 60$ (years). Simulate and draw the graphs of each transition probability for different values of x, say $x \in [0, 100]$.

(ii) Now assume that the transition rates are given by the Gompertz-Makeham model as follows

$$\mu_{*\diamond}(t) = a_1 + b_1 \exp(c_1 t) \quad \mu_{*\dagger}(t) = a_2 + b_2 \exp(c_2 t) \quad \mu_{\diamond\dagger}(t) = \mu_{*\dagger}(t),$$

where $a_1 = 4 \cdot 10^{-4}$, $b_1 = 3.4674 \cdot 10^{-6}$, $c_1 = 0.138155$, $a_2 = 5 \cdot 10^{-4}$, $b_2 = 7.5858 \cdot 10^{-5}$ and $c_2 = 0.087498$. Calculate $p_{**}(x, x+10)$ and $p_{*\diamond}(x, x+10)$ for $x = 60$ (years). Simulate and draw the graphs of each transition probability for different values of x, say $x \in [0, 100]$.