STK4500: Life Insurance and Finance

Exercise list 10: Solutions

Exercise 10.1

Use the integration by parts formula or Itô's formula to verify that a portfolio strategy $\phi_t = (\phi_t^0, \phi_t^1)$, $t \geqslant 0$ is self-financing if, and only if

$$\widetilde{V}_t(\phi) = V_0(\phi) + \int_0^t \phi_s^1 d\widetilde{S}(s), \quad t \geqslant 0,$$

where $\widetilde{S}(t) := e^{-rt} S(t)$ is the discounted (Black-Scholes) stock price and $\widetilde{V}_t(\phi) := e^{-rt} V_t(\phi)$ the discounted portfolio value with respect to ϕ_t.

Exercise 10.2 (Markov property of Black-Scholes stock prices)

(i) Use Exercise 9.1 from List 9 to show that

$$S(t) = x \exp\left(\left(r - \frac{\sigma^2}{2}\right) t + \sigma \widetilde{B}_t\right),$$

where $\widetilde{B}_t$, $0 \leqslant t \leqslant T$ is a Brownian motion w.r.t. the equivalent martingale measure $\mathbb{Q}$.

(ii) Use (i) and Exercise 2 from List 9 to find a formula for the present value at time t of the contingent claim

$$X = \max\{N(T)S(T), G(T)\},$$

where $N(T)$ is the (deterministic) number of shares invested in the stock index (or fund) and $G(T)$ a guaranteed payment.

Exercise 10.3

Consider a 10-year unit-linked pure endowment insurance (or guaranteed minimum maturity benefit) issued to a life aged 60 with a single premium of $\pi_0 = 10\,000\$$. An initial expense deduction of 3% is charged and the rest of the premium is invested in an equity fund whose dynamics S_t of its values over time is described by the Black-Scholes model in Exercise 9.1 from List 9 or Exercise 10.2 above, with $S_0 = 1$. Further, an annual management charge of 0.5% is deducted from the fund at the beginning of each year with the exception of the first year. The maturity benefit (or endowment amount) is given by the maximum of the investment at maturity and the premium π_0.

Assume

(i) Makeham's law

$$\mu_{*\dagger}(t) = A + Bc^t,$$

with $A = 0.0001$, $B = 0.00035$ and $c = 1.075$. Or if you want, you can use Norwegian mortality data from https://www.ssb.no/dode (Table 2) and using the data to estimate A, B and c.

(ii) Risk free rate of interest $r = 5\%$ per year, continuously compounded.

(iii) Volatility $\sigma = 25\%$ per year of S_t.

Calculate the prospective reserve of the maturity benefit at the initial time of the contract.
Solution: The policy function that defines entirely this contract is given by

$$a_*(t) = \begin{cases} 0, \text{ if } 0 \leqslant t < 0, \\ -t\pi, \text{ if } \leqslant 0 < t < 10, \\ -10\pi + C(T), \text{ if } t \geqslant 10 \end{cases}$$

Here: π premium payment, $C(T)$ endowment given by

$$C(T) \triangleq \max\left\{(1 - 0.03) \cdot \pi_0 \cdot S_{10}(1 - 0.005)^9, \pi_0\right\},$$

where 0.03 is the initial expense deduction and 0.005 stands for the the management charges.
Using equation (9.15) from the lecture notes, we have

$$V_{\mathcal{F}}^{+}(t, A) = \pi_t(T)p_{**}(x + t, x + T) - \int_t^T e^{-r(s-t)}p_{**}(x + t, x + s)\pi ds, \text{ if } X_t = *.$$

Here, $x = 60$ is the age of the insured when entering the contract and $\pi_t(T) \triangleq \text{ClaimValue}_t = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}C(T)\middle|\mathcal{G}_t\right]$ (see formula (9.12) in the notes).
Set $\pi = 0$ in the formula above. The prospective reserve for the endowment payment:

$$V_{\mathcal{F}}^{+}(t, A) = \pi_t(T)p_{**}(t, T), \text{ if } X_t = *. \tag{0.1}$$

At time $t = 0$ (when the insured is 60),

$$V_{\mathcal{F}}^{+}(0, A) = \pi_0(T)\underbrace{p_{**}(60, 70)}_{0.673958}.$$

From Exercise 2, above, (similar to the Black-Scholes formula), we have

$$\pi_t(T) = G(T)\exp(-r(r - t))\Phi(-d_2(T - t)) + S_tN(T)\Phi(d_t(T - t)), \tag{0.2}$$

with

$$\Phi(y) \triangleq \frac{1}{2\pi}\int_{-\infty}^{y} \exp\left(-\frac{z^2}{2}\right) dz,$$

$$d_1(s) \triangleq \frac{\log\left(\frac{N(T)S_t}{G(T)}\right) + \left(r + \frac{1}{2}\sigma^2\right)s}{\sigma\sqrt{s}}, \quad d_2(s) \triangleq \frac{\log\left(\frac{N(T)S_t}{G(T)}\right) + \left(r - \frac{1}{2}\sigma^2\right)s}{\sigma\sqrt{s}}.$$

We have $S_0 = 1$, $N(T) = \underbrace{(1-0.03)(1-0.005)^9}_{0.927213}\pi_0$, $G(T) = P$, $T = 10$, $r = 0.05$, $\sigma = 0.25$ and $t = 0$. Then

$$\pi_t(T) = 1.033488\pi_0 = 10\,334.88\$$$

Hence

$$V^+_{\mathcal{F}}(0, A) = 6\,965.28\$$$

Exercise 10.4

Suppose the insurance contract in Exercise 10.3 is still in force 6 years after its issue. Calculate the prospective reserve of the maturity benefit at time $t = 6$ years in the case that the fund value since its purchase

Solution: From (0.1) in the previous exercise we know that

$$V^+_{\mathcal{F}}(6, A) = \pi_6(T)\underbrace{p_{**}(66, 70)}_{=0.824935}, \quad X_6 = *.$$

(i) increased by 45%, and

Solution: $S_0 = 1$ increased by 45% implies $S_6 = 1.45$, then by (0.2) when $t = 6$ we have

$$V^+_{\mathcal{F}}(6, A) = 11\,449.83\$, \quad if X_6 = *$$

(ii) increased by 5%.

Solution: $S_0 = 1$ increased by 5% implies $S_6 = 1.05$, then by (0.2) when $t = 6$ we have

$$V^+_{\mathcal{F}}(6, A) = 8\,936.74\$, \quad if X_6 = *$$

Exercise 5 (Black-Scholes partial differential equation)

Consider the European call option $X = \max\{0, S_T - K\}$ where $K > 0$ is the strike price and $S = \{S_t, t \in [0, T]\}$ the stock price process described by the Black-Scholes SDE

$$S_t = x + \int_0^t S(s)\mu ds + \int_0^t S(s)\sigma dB_s,$$

for constants $\mu \in \mathbb{R}$ and $\sigma > 0$. Use Exercise 9.3 from List 9 to write the fair value of X at time t as

$$\text{ClaimValue}_t = v(t, S_t)$$

for a function v. Show that v solves the partial differential equation

$$-\frac{\partial v}{\partial t} + rv = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + rx\frac{\partial v}{\partial x}, \quad t \in [0, T], x \in (0, \infty),$$

$$v(T, x) = \max\{0, x - K\}, \quad x \in [0, \infty).$$

Solution: We know by Theorem 9.12 (pricing formula) that

$$ClaimValue_t = \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}X|\mathcal{G}_t] = V_t(\phi)$$

where $\mathbb{Q}$ is the risk neutral probability. Then

$$\widetilde{V}_t(\phi) \triangleq e^{-rt}V_t(\phi) = e^{-rt}ClaimValue_t = e^{-rt}v(t, S_t),$$

where (by Exercise 9.2 in List 9)

$$v(t,x) = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}g\left(xe^{r(T-t)}e^{\sigma(\widetilde{B}_T-\widetilde{B}_t)-\frac{\sigma^2}{2}(T-t)}\right)\right],$$

where $\widetilde{B}_t$, $0 \leqslant t \leqslant T$ is the "new" Brownian motion under $\mathbb{Q}$ and $g(y) \triangleq \max\{0, x-K\}$.

Define

$$\widetilde{v}(t,x) = e^{-rt}v(t, xe^{rt}).$$

Then

$$\widetilde{V}_t(\phi) = \widetilde{v}(t, \widetilde{S}(t)).$$

Using Itô's formula we see (Exercise 9.1, List 9) that

$$\widetilde{S}(t) = \widetilde{S}(0) + \int_0^t (\mu - r)\widetilde{S}(s)ds + \int_0^t \sigma\widetilde{S}(s)dB_s.$$

Then Itô's formula applied to $\widetilde{S}(t)$ and the function $\widetilde{v}(t,x)$ under $\mathbb{P}$ gives

$$\begin{aligned}\widetilde{v}(t, \widetilde{S}(t)) = &\widetilde{v}(0, \widetilde{S}(0)) + \int_0^t \frac{\partial}{\partial t}\widetilde{v}(s, \widetilde{S}(s))ds + \int_0^t \frac{\partial}{\partial x}\widetilde{v}(s, \widetilde{S}(s))(\mu - r)\widetilde{S}(s)ds \\ &+ \int_0^t \frac{\partial}{\partial x}\widetilde{v}(s, \widetilde{S}(s))\sigma\widetilde{S}(s)dB_s.\end{aligned}$$

On the other hand, by Exercise 10.1 above we also know that

$$\widetilde{V}_t(\phi) = V_0(\phi) + \int_0^t \phi_s^{(1)}d\widetilde{S}(s) = V_0(\phi) + \int_0^t \phi_s^{(1)}(\mu - r)\widetilde{S}(s)ds + \int_0^t \phi_s^{(1)}\sigma\widetilde{S}(s)dB_s.$$

Now use the following fact (see lecture notes):

$$X_t = X_0 + \int_0^t u(s)ds + \int_0^t v(s)dB_s, \quad Y_t = Y_0 + \int_0^t u^*(s)ds + \int_0^t v^*(s)dB_s$$

with $X_t = Y_t$, $0 \leqslant t \leqslant T$ then

$$u = u^*, \quad v = v^*.$$

The above fact allows us to conclude that

$$\phi_s^{(1)} \underbrace{\sigma\widetilde{S}(s)}_{>0} = \frac{\partial}{\partial x}\widetilde{v}(t, \widetilde{S}(s))\sigma\widetilde{S}(s),$$

if, and only if

$$\phi_s^{(1)} = \frac{\partial}{\partial x}\widetilde{v}(t, \widetilde{S}(s)) = \frac{\partial}{\partial x}v(t, S(s)).$$

On the other hand, we get, by comparing ds-terms that

$$\frac{\partial}{\partial s}\widetilde{v}(t,\widetilde{S}(s)) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\widetilde{v}(t,\widetilde{S}(s))\sigma^2(\widetilde{S}(s))^2 = 0.$$

By the definition of $\widetilde{v}(s,x)$ we get

$$\frac{\partial}{\partial s}v(s,S(s)) + rS(s)\frac{\partial}{\partial x}v(s,S(s)) + \frac{1}{2}\sigma^2 S^2(s)\frac{\partial^2}{\partial x^2}v(s,S(s)) = rv(s,S(s)).$$

We can now choose $S(s) = x$ (since B_t has a probability density) and obtain the Black-Scholes partial differential equation:

$$\frac{\partial}{\partial s}v(s,x) + rx\frac{\partial}{\partial x}v(s,x) + \frac{1}{2}\sigma^2 x^2\frac{\partial^2}{\partial x^2}v(s,x) = rv(s,x)$$

with boundary condition $v(T,x) = g(x)$.