STK4500: Life Insurance and Finance

Exercise list 1: Solutions

Exercise 1

Let S_n be the price of a stock at day $n = 0, 1, 2, \dots$ ($n = 0$ present month). Suppose that $S_n \in \{0\$, 10\$, 15\$\}$ for all n and that the current stock price is 10\$, i.e. $S_0 = 10$ \$. Further assume that stock prices $\{S_n\}_{n\in\mathbb{N}}$ are modeled by a Markov chain with transition probability matrix P given by

$$P = \begin{pmatrix} p_{0\$\ 0\$} & p_{0\$\ 10\$} & p_{0\$\ 15\$} \\ p_{10\$\ 0\$} & p_{10\$\ 10\$} & p_{10\$\ 15\$} \\ p_{15\$\ 0\$} & p_{15\$\ 10\$} & p_{15\$\ 15\$} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0.02 & 0.68 & 0.3 \\ 0.01 & 0.64 & 0.35 \end{pmatrix}$$

Compute $P(S_5 = 15\ \$ | S_0 = 10\ \$)$. Compute the expected number of months until bankruptcy (check theory on Markov chains). Compute both quantities using R or any other programming language.

Solution: Let $P(s,t) = \{p_{ij}(s,t)\}_{i,j\in\{0\$,10\$,15\$\}}$. This Markov chain is (time) homogeneous, so $P(s,t) = P$. Using Chapman-Kolmogorov equation repeatedly we can write

$$P(0,5) = P(0,1)P(1,2)\cdots P(4,5) = P(0,1)^5 = P^5 = \begin{pmatrix} 1 & 0 & 0 \\ 0.0843 & 0.6213 & 0.2944 \\ 0.0745 & 0.628 & 0.2975 \end{pmatrix}.$$

Hence,

$$p_{10\$\ 15\$}(0,5) = P(X_5 = 10\$ | X_0 = 10\$) = 0.2944$$

This Markov chain is absorbing. This means that, eventually, we will enter state $\{0\$\}$ and never leave. For theory on Markov chains it is possible to compute the expected number of transitions until absorption. To do so, we must identify the part of the matrix which is transient, i.e.

$$P = \begin{pmatrix} Q & R \\ \mathbf{0} & I \end{pmatrix}$$

In our case,

$$Q = \begin{pmatrix} 0.68 & 0.3 \\ 0.64 & 0.35 \end{pmatrix}.$$

The fundamental matrix is given by

$$N = (I_{2\times 2} - Q)^{-1} = \begin{pmatrix} 40.625 & 18.75 \\ 40 & 20 \end{pmatrix}.$$

The (i, j) entry of matrix N is the expected number of times the chain is in state j, given that the chain started in state i. The expected total number of transitions before reaching 0$ starting from 10$ and 15$ respectively is given by

$$\begin{pmatrix} 40.625 & 18.75 \\ 40 & 20 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 59.375 \\ 60 \end{pmatrix}.$$

See this wikipedia article for a summary of the theory: https://en.wikipedia.org/wiki/Absorbing_Markov_chain

R-code used:

```
Q <- matrix(c(0.68, 0.3, 0.64, 0.35), nrow=2, byrow=TRUE)
I <- matrix(c(1, 0, 0, 1), nrow=2, byrow=TRUE)
one <- matrix(c(1, 1), nrow=2, byrow=TRUE)
N <- solve(I-Q)
N%*%one
```

Exercise 2

Let $N = \{N(t), t \geqslant 0\}$ be a Poisson process with intensity $\lambda > 0$ on a probability space $(\Omega, \mathcal{F}, P)$, that is, $N = \{N(t), t \geqslant 0\}$

(i) has right-continuous sample paths with existing left limits (càdlàg paths)

(ii) starts at 0

(iii) has independent and stationary increments

(iv) $N(t)$ is Poisson distributed with parameter λt, that is

$$P(N(t) = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}, \quad n = 0, 1, \ldots$$

Show that the compensated Poisson process is a martingale with respect to the filtration $\mathcal{F}_t = \sigma(N(s), 0 \leqslant s \leqslant t)$

Solution: The compensated Poisson process is defined as $M(t) = N(t) - \lambda t$. This process is P-integrable, i.e. $E[|M(t)|] = E[|N(t) - \lambda t|] \leqslant 2\lambda t < \infty$ for every $t \geqslant 0$. $M(t)$ is $\mathcal{F}_t$ measurable by definition since $M(t) = f_t(N(t))$ for $f_t(x) = x - \lambda t$ and f_t is a continuous bijection. Finally, M satisfies the martingale property

$$E[M(t)|\lambda_s] = E[N(t)|\mathcal{F}_s] - \lambda t = E[N(t) - N(s)|\mathcal{F}_s] + N(s) - \lambda t = \lambda(t-s) + N(s) - \lambda t = M(s).$$

Exercise 3 (De Moivre's martingale)

Suppose a coin is unfair and denote by p the probability heads and by $q = 1 - p$ the probability of tails. Let $X_{n+1} = X_n \pm 1$ with $+$ in case of heads obtaining heads and $-$ in case of obtaining tails. Let $Y_n = (q/p)^{X_n}$, $n \geqslant 1$. Show that the process $Y = \{Y_n, n \geqslant 1\}$ is a martingale w.r.t.

the filtration generated by $X_1, \dots, X_n$, i.e. prove that Y is adapted to the filtration generated by $X_1, \dots, X_n$, $E[|Y_n|] \leqslant \infty$ for every $n \geqslant 1$ and finally $E[Y_{n+1}|\mathcal{F}_n] = Y_n$ for every $n \geqslant 1$

Solution: The process is P-integrable

$$E[|Y_n|] = E[E[(q/p)^{X_n}|X_{n-1}]] = E[(q/p)^{X_{n-1}+1}p+(q/p)^{X_{n-1}-1}q] = pE[Y_{n-1}]+qE[Y_{n-1}] = E[Y_{n-1}].$$

This implies that $E[|Y_n|] = E[Y_0] = E[(q/p)^{X_0}]$ so we need to assume that X_0 is such that $E[(q/p)^{X_0}] < \infty$. Y_n is indeed measurable w.r.t. $\sigma(X_1, \dots, X_n)$ since $Y_n = f(X_n)$ with $f(x) = (q/p)^x$ and this function is injective, which implies that $\sigma(Y_n) = \sigma(X_n) \subseteq \sigma(X_1, \dots, X_n)$. Finally, Y satisfies the martingale property

$$E[Y_{n+1}|\sigma(X_1, \dots, X_n)] = E[(q/p)^{X_{n+1}}|\sigma(X_1, \dots, X_n)] = (q/p)^{X_n+1}p + (q/p)^{X_n-1}q = Y_n.$$

Exercise 4

Let X_0 be a random variable with values in the countable state space S. Let $Y_1, Y_2, \dots$ be a sequence of i.i.d. with uniform distribution on the interval $[0, 1]$. Consider a given function

$$G : S \times [0, 1] \to S$$

and define recursively

$$X_{n+1} = G(X_n, Y_{n+1}), \quad n \geqslant 0.$$

Show that $X = \{X_n\}_{n\geqslant 0}$ is a Markov chain and express its transition matrix in terms of G. Can all (time-homogeneous) Markov chains with index set $J = \{0, 1, 2, \dots\}$ be realized in this way? How would you simulate a Markov chain? Try a function G of your own choice and simulate X.

Solution: In order to show that $X = \{X_n\}_{n\geqslant 0}$ is a homogenenous Markov chain it is enough to verify

$$P(X_{n+1} = i_{n+1}|X_0 = i_0 \dots, X_n = i_n) = P(X_{n+1} = i_{n+1}|X_n = i_n) = p_{i_n,i_{n+1}}.$$

We know that

$$\begin{aligned} P(X_{n+1} &= i_{n+1}|X_0 = i_0, X_1 = i_1 \dots, X_n = i_n) = \\ &= P(G(X_n, Y_{n+1}) = i_{n+1}|X_0 = i_0, X_1 = i_1 \dots, X_n = i_n) \\ &= P(G(i_n, Y_{n+1}) = i_{n+1}|X_0 = i_0, G(X_0, Y_1) \dots, G(X_{n-1}, Y_n) = i_n) \end{aligned}$$

Observe that the event $G(i_n, Y_{n+1}) = i_{n+1}$ depends on Y_{n+1} and the event $X_0 = i_0, G(X_0, Y_1) \dots, G(X_{n-1}, Y_n$ i_n depends only on $X_0, \dots, X_n$ and Y_n, hence they are independent and we can conclude that

$$P(X_{n+1} = i_{n+1}|X_0 = i_0, X_1 = i_1 \dots, Xc_n = i_n) = P(G(i_n, Y_{n+1}) = i_{n+1}) = P(G(i_n, Y_1) = i_{n+1})$$

where the latter follows since Y_n's are identically distributed.

On the other hand,

$$P(X_{n+1} = i_{n+1}|X_n = i_n) = P(G(X_n, Y_{n+1}) = i_{n+1}|X_n = i_n) = P(G(i_n, Y_1) = i_{n+1}|X_n = i_n)$$

and this implies that $\{X_n\}_{n\geqslant 1}$ is a Markov chain.

Assume now that $S=\{1,2,\dots\}$. Choose $G: S\times[0,1]\to S$ by

$$G(i,y) := \sum_{j\in S} j\mathbf{1}_{\{F^i_{j-1}<y\leqslant F^i_j\}}, \tag{0.1}$$

where $F^i_0=0$ and $F^i_k=\sum_{j=1}^k p_{ij}$, $k\geqslant 1$.

Take $Y_1\sim U(0,1)$, then the distribution of $Z_n := G(n,Y_1)$ is given by p_{nj}, $j\geqslant 1$. Indeed, $P(Z_n=j)=p_{nj}$, $j\geqslant 1$ due to the definition of G (categorical variable).

For G defined as in (0.1) and given transition probabilities p_{ij}, $i,j\in S$, $\{X_n\}_{n\geqslant 1}$ is a hom. Markov chain with such transition probabilities. This gives the following procedure for the simulation of hom. Markov chains with transition probabilities p_{ij}, $i,j\in S$:

Draw $y_1,\dots,y_n$ from the random number generator of $U(0,1)$ and draw $x_0\sim X_0$ from the random number generator of X_0.

Output: $x_0, x_1 := G(x_0,y_1),\dots,\ x_n=G(x_{n-1},y_n)$. Repetition of this procedure gives other simulated paths of $\{X_n\}_{n\geqslant 1}$.

Exercise 5 (Permanent disability model)

Assume in this model that the state of the insured $X_t\in S$ is modeled by a regular Markov chain with state space $S=\{*,\diamond,\dagger\}$ where $*=$"active", $\diamond=$"disabled" and $\dagger=$"dead". The permanent disability model for an insurance policy provides some of the following benefits: (i) an annuity while permanently disabled, (ii) a lump sum on becoming permanently disabled and (iii) a lump sum on death. Premiums are paid while the insured is healthy ,i.e. active. An important feature of this model is that disablement is permanent, that is $p_{\diamond *}(s,t)=0$, $s\leqslant t$.

(i) Suppose the transition rates for this model are all constants and given by

$$\mu_{*\diamond}(t)=0.0279 \quad \mu_{*\dagger}(t)=0.0229 \quad \mu_{\diamond\dagger}(t)=\mu_{*\dagger}(t).$$

Calculate $p_{**}(x,x+10)$ and $p_{*\diamond}(x,x+10)$ for $x=60$ (years). Simulate and draw the graphs of each transition probability for different values of x, say $x\in[0,100]$.

(ii) Now assume that the transition rates are given by the Gompertz-Makeham model as follows

$$\mu_{*\diamond}(t)=a_1+b_1\exp(c_1t) \quad \mu_{*\dagger}(t)=a_2+b_2\exp(c_2t) \quad \mu_{\diamond\dagger}(t)=\mu_{*\dagger}(t),$$

where $a_1=4\cdot 10^{-4}$, $b_1=3.4674\cdot 10^{-6}$, $c_1=0.138155$, $a_2=5\cdot 10^{-4}$, $b_2=7.5858\cdot 10^{-5}$ and $c_2=0.087498$. Calculate $p_{**}(x,x+10)$ and $p_{*\diamond}(x,x+10)$ for $x=60$ (years). Simulate and draw the graphs of each transition probability for different values of x, say $x\in[0,100]$.

Solution: Idea: forward Kolmogorov equation: $\partial_t P(s,t)=P(s,t)\Lambda(t)$.

$$\frac{d}{dt}p_{**}(s,t)=p_{**}(s,t)\mu_{**}(t)+p_{*\diamond}(s,t)\mu_{\diamond *}(t)+p_{*\dagger}(s,t)\mu_{\dagger *}(t).$$

Because of the assumptions of this insurance we see that $\mu_{\diamond *}(t) = 0$ and $\mu_{\dagger *}(t) = 0$. The solution of the above equation is then given by

$$p_{**}(s,t) = \exp\left(\int_s^t \mu_{**}(u)du\right).$$

Also, $\mu_{**}(t) = -\mu_*(t)$ and $\mu_{**}(t) + \mu_{*\diamond}(t) + \mu_{*\dagger}(t) = 0$, so

$$\mu_*(t) = \mu_{*\diamond}(t) + \mu_{*\dagger}(t).$$

Altogether,

$$\boxed{p_{**}(s,t) = \exp\left(-\int_s^t \mu_{*\diamond}(u)du - \int_s^t \mu_{*\dagger}(u)du\right)}$$

Now the transition probability from state $*$ to $\diamond$ is given by the solution of

$$\frac{d}{dt}p_{*\diamond}(s,t) = p_{**}(s,t)\mu_{*\diamond}(t) + p_{*\diamond}(s,t)\mu_{\diamond\diamond}(t) + p_{*\dagger}(s,t)\mu_{\dagger\diamond}(t).$$

Simplifying,

$$\frac{d}{dt}p_{*\diamond}(s,t) = p_{**}(s,t)\mu_{*\diamond}(t) + p_{*\diamond}(s,t)\mu_{\diamond\diamond}(t).$$

Observe that $\mu_{\diamond *}(t) = 0$ and hence $\mu_{\diamond\diamond}(t) = -\mu_{\diamond\dagger}(t)$. Hence,

$$\frac{d}{dt}p_{*\diamond}(s,t) + p_{*\diamond}(s,t)\mu_{\diamond\dagger}(t) = p_{**}(s,t)\mu_{*\diamond}(t).$$

Observe that the left-hand side is almost the derivative of $p_{*\diamond}(s,t)e^{\int_s^t \mu_{\diamond\dagger}(u)du}$ w.r.t. t, we just need to multiply with the factor $e^{\int_s^t \mu_{*\diamond}(u)du}$. So,

$$e^{\int_s^t \mu_{\diamond\dagger}(u)du}\frac{d}{dt}p_{*\diamond}(s,t) + e^{\int_s^t \mu_{\diamond\dagger}(u)du}p_{*\diamond}(s,t)\mu_{\diamond\dagger}(t) = p_{**}(s,t)\mu_{*\diamond}(t)e^{\int_s^t \mu_{\diamond\dagger}(u)du}.$$

That is

$$\frac{d}{dt}\left[e^{\int_s^t \mu_{\diamond\dagger}(u)du}p_{*\diamond}(s,t)\right] = p_{**}(s,t)\mu_{*\diamond}(t)e^{\int_s^t \mu_{\diamond\dagger}(u)du}.$$

Integrating we have

$$e^{\int_s^t \mu_{\diamond\dagger}(u)du}p_{*\diamond}(s,t) = \int_s^t p_{**}(s,r)\mu_{*\diamond}(r)e^{\int_s^r \mu_{\diamond\dagger}(u)du}dr$$

or rather

$$\boxed{p_{*\diamond}(s,t) = \int_s^t p_{**}(s,r)\mu_{*\diamond}(r)e^{-\int_r^t \mu_{\diamond\dagger}(u)du}dr}$$

Because of the assumptions of this insurance we also see that

$$p_{**}(s,t) = \overline{p}_{**}(s,t) := P(X_u = * \text{ for all } u \in [s,t] | X_s = *)$$

and

$$p_{\diamond\diamond}(s,t) = \overline{p}_{\diamond\diamond}(s,t),$$

since the probabilities for jumping back to state $*$ or $\diamond$ are zero.

We also know from the lectures that we have

$$\overline{p}_{jj}(s,t) = \exp\left(-\sum_{k\neq j}\int_s^t \mu_{jk}(u)du\right)$$

and setting $j = *$ we also obtain

$$p_{**}(s,t) = \overline{p}_{**}(s,t) = \exp\left(-\int_s^t \mu_{*\diamond}(u)du - \int_s^t \mu_{*\dagger}(u)du\right)$$

(i) Since the transition rates are constant, expressions simply a lot. We have

$$p_{**}(s,t) = \exp\left(-0.0508(t-s)\right) \Rightarrow p_{**}(60,70) = e^{-0.0508\cdot 10} = 0.6016978$$

and

$$p_{*\diamond}(s,t) = \int_s^t p_{**}(s,r)\mu_{*\diamond}(r)e^{-\int_r^t \mu_{\diamond\dagger}(u)du}dr \Rightarrow p_{*\diamond}(s,t) = \frac{-0.0279}{0.0229}\left(e^{-0.0508(t-s)} - e^{-0.0279(t-s)}\right)$$

and hence

$$p_{*\diamond}(60,70) = \frac{-0.0279}{0.0229}\left(e^{-0.0508\cdot 10} - e^{-0.0279\cdot 10}\right) = 0.1886505$$

The graphs are constant since $p_{**}(x,x+10) = p_{**}(60,60+10)$ and similar for $p_{*\diamond}(x,x+10)$. This is due to the fact that transition rates are time-independent. So, the probability of dying in 10 years is unchanged. This is obviously not a realistic model. The next model is more realistic since transition rates increase in time and hence transition probabilities too.

(ii) Using the new rates we have

$$p_{**}(s,t) = e^{-a_1(t-s)-\frac{b_1}{c_1}(e^{c_1t}-e^{c_1s})-a_2(t-s)-\frac{b_2}{c_2}(e^{c_2t}-e^{c_2s})} \Rightarrow p_{**}(60,70) = 0.5839526$$

For $p_{*\diamond}(s,t)$ we find that

$$p_{*\diamond}(s,t) = e^{-a_1(t-s)-\frac{b_1}{c_1}(e^{c_1t}-e^{c_1s}))}e^{a_2s+\frac{b_2}{c_2}e^{c_2s}}\int_s^t (a_1+b_1e^{c_1r})e^{-a_2r-\frac{b_2}{c_2}e^{c_2r}}dr$$

and integrating numerically we have

$$p_{*\diamond}(60,70) = 0.1969163$$