STK4500: Life Insurance and Finance

Exercise list 2: Solutions

Exercise 2.1 (Mortality basis from The Financial Supervisory Authority of Norway)

Finanstilsynet (The Financial Supervisory Authority of Norway) published the following letter in 2013 with the mortality basis Norwegian insurances companies have to comply with. The document is popularly known as $K2013$. Go to K2013 to download it. There you will find different functions for $\mu(x,t)$ according to gender and risk. Here, x is the age of the insured and t is the calendar year. As you know, mortality changes from year to year in the sense that, a person who is x today, say $t = 2021$ will not have the same mortality as a person who is x years old next year $t = 2022$.

(a) Consider two states $S = \{*, \dagger\}$ and $\mu(t) = \mu_{*\dagger}(t)$, $t \geqslant 0$ the mortality rate. Use Kolmogorov's equation to show that the probability of an x year-old will survive s more units of time, given that they are alive at $x+t$ is given by

$$p_{**}(x+t, x+s) = \exp\left(-\int_t^s \mu(x+u)du\right), \quad s,t \geqslant 0, \quad s \geqslant t.$$

Solution:

Forward's Kolmogorov equation for $t \mapsto p_{**}(s,t)$ has initial condition $p_{**}(s,s) = 1$ and satisfies for every $t \geqslant s$,

$$\frac{d}{dt}p_{**}(s,t) = p_{**}(s,t)\mu_{**}(t) + p_{*\dagger}(s,t)\underbrace{\mu_{\dagger *}(t)}_{=0}.$$

See Theorem 1.15 in the lecture notes (page 5). From the equation above, it is readily checked that

$$p_{**}(s,t) = \exp\left(\int_s^t \mu_{**}(u)du\right), \quad t \geqslant s.$$

Recall that $\mu_{**}(t) + \mu_{*\dagger}(t) = 0$, so $\mu_{**}(t) = -\mu_{*\dagger}(t)$ and we obtain

$$p_{**}(s,t) = \exp\left(-\int_s^t \mu_{*\dagger}(u)du\right), \quad t \geqslant s.$$

The probability that a person who is alive at age $x+t$ will survive until $x+s$, $s \geqslant t$is given by

$$\mathbb{P}[X_{x+s} = * | X_{x+t} = *] = p_{**}(x+t, x+s) = \exp\left(-\int_{x+t}^{x+s} \mu_{*\dagger}(u)du\right) = \exp\left(-\int_{t}^{s} \mu_{*\dagger}(x+u)du\right),$$

where the latter step follows by a linear change of variables.

(b) If we take $\mu(t) = \mu_{Kol}(x,t)$ where μ_{Kol} are the mortality rates from Finanstilsynet, then for a life aged x in year $Y \geqslant 2013$ we have

$$p_{**}(x+t, x+s) = \exp\left(-\int_{t}^{s} \mu_{Kol}(x+u, Y+u)du\right), \quad s,t \geqslant 0, \quad s \geqslant t. \tag{0.1}$$

In particular, the probability of surviving one more year given that one is x years old in 2021 is given by

$$p_{**}(x, x+1) = \exp\left(-\int_{0}^{1} \mu_{Kol}(x+u, 2021+u)du\right).$$

Use Taylor's formula (of order one) to prove the (rather very rough) approximation

$$p_{**}(x, x+1) \approx \exp\left(-\mu_{Kol}(x, 2021)\right).$$

Solution: Taylor's formula of order one for a differentiable function f is given by

$$f(x+h) = f(x) + f'(x)h + O(h^2), \quad x \in \mathbb{R}.$$

In particular, the approximation around zero (also known as Mc Laurin formula) is

$$f(h) = f(0) + f'(0)h + O(h^2).$$

Define $f(z) = \int_0^z \mu_{Kol}(x+u, 2021+u)du$ then $f'(z) = \mu_{Kol}(x+z, 2021+z)$ by the fundamental theorem of calculus (since μ_{Kol} is continuous on $[0,z]$ for every $z \in \mathbb{R}$ and in particular Riemann-integrable too). Then $f(h) = \int_0^h \mu_{Kol}(x+u, 2021+u)du$, $f(0) = 0$ and $f'(0) = \mu_{Kol}(x, 2021)$. Then applying the formula above we have

$$\int_0^h \mu_{Kol}(x+u, 2021+u)du = \mu_{Kol}(x, 2021)h + O(h^2).$$

If $h = 1$ then

$$\int_0^1 \mu_{Kol}(x+u, 2021+u)du = \mu_{Kol}(x, 2021) + O(1),$$

and the formula follows.

If one, in addition, uses the rather rough (Taylor) approximation for the exponential function as well $e^x \approx 1+x$ then one can approximate the survival probability as

$$p_{**}(x, x+1) \approx 1 - \mu_{Kol}(x, 2021).$$

I have seen students using the above approximation (which is a combination of two approximations). Be careful! I would not recommend to use such a rough approximation when dealing with risk.

(c) In general, we prefer a more accurate integration method to find p_{**}. Use Riemann sums, trapezoidal rule and Simpson's method for finding an approximate value for

$$\int_a^b f(t)dt,$$

for a Riemann integrable function f on $[a, b]$.

Apply this to $K2013$ with mortality risk and compute

$$p_{**}(x, x+t), \quad t \in \{0, 10, 20, 30, 40, 50\},$$

where x is your age.

Plot the function $t \mapsto p_{**}(x, x+t)$ where x is your age. Remember that you have to choose μ_{Kol} according to your gender.

Solution:

Riemann sums

We approximate the function f by rectangles (from the left point, right or middle) and then sum the areas of each rectangle. Take the uniform partition of $[a, b]$, $t_i = a + ih$, $i = 0, \ldots, n$ where h is such that $h = \frac{b-a}{n}$. Then

$$\int_a^b f(t)dt \approx \frac{1}{n} \sum_{i=0}^{n-1} f(t_i) \quad \text{(left Riemann sum)}$$

$$\int_a^b f(t)dt \approx \frac{1}{n} \sum_{i=0}^{n-1} f\left(\frac{t_i + t_{i+1}}{2}\right) \quad \text{(middle Riemann sum)}$$

$$\int_a^b f(t)dt \approx \frac{1}{n} \sum_{i=1}^{n} f(t_i) \quad \text{(right Riemann sum)}$$

Trapezoidal rule

We approximation the area of the function on $[t_i, t_{i+1}]$, $i = 0, \ldots, n-1$ by the trapezoid limited by $t_i, t_{i+1}, f(t_i)$ and $f(t_{i+1})$. Thus

$$\int_a^b f(t)dt \approx \frac{1}{n} \sum_{i=0}^{n-1} \frac{f(t_i) + f(t_{i+1})}{2} \quad \text{(Trapezoidal rule)}$$

Simpson's rule

It is based on approximating each piece of the function on $[t_i, t_{i+1}]$ by a parabola. Divide the interval $[a, b]$ into equal pieces $t_i = a + ih$, $i = 0, \dots, n$ where h is such that $h = \frac{b-a}{n}$ and n is even. Then

$$\int_a^b f(t)dt \approx \frac{h}{3}\left[f(a) + 2\sum_{i=1}^{\frac{n}{2}-1} f(t_{2i}) + 4\sum_{i=1}^{\frac{n}{2}} f(t_{2i-1}) + f(b)\right] \qquad \text{(Simpson's rule)}$$

(d) Write an R-code which generates random lives with the mortalities given by Finanstilsynet. Plot many life times in a histogram. Compute the empirical descriptive statistics and check that they are close to the theoretical ones.

Hint: Fix x and let T be the remaning life time of an x year old person. Then the total life time is $T_x \triangleq x + T$. What is the distribution (function) of T_x?. When you detect the distribution function of T_x, use the inverse transform sampling method to simulate values from T_x. The inverse transform method is based on the following result: if Z is a random variable with distribution function F_Z then $F_Z(Z)$ is uniformly distributed on $[0, 1]$.

<u>Solution:</u>

Let x be the age of someone in 2021. What is the distribution of the remaining life time, say T_x? Observe the following: the probability, given that the are alive at age x, of being alive for t more units of age is

$$p_{**}(x, x+t) = \mathbb{P}[X_{x+t} = *|X_x = *] = \mathbb{P}[T_x > t|X_x = *].$$

Now, observe that

$$\mathbb{P}[T_x \leqslant t|X_x = *] = 1 - \mathbb{P}[T_x > t|X_x = *] = 1 - p_{**}(x, x+t) = p_{*\dagger}(x, x+t)$$

Hence, the distribution function of the remaining life T_x time of an x year old person is given by the (distribution) function

$$F_x(t) \triangleq p_{*\dagger}(x, x+t).$$

Observe that $t \in [0, \infty)$ and it is indeed a distribution function:

- F_x is increasing in t.
- $\lim_{\substack{t\to 0\\ t>0}} F_x(t) = 0$, $\lim_{t\to\infty} F_x(t) = 1$.

So,

$$T_x \sim F_x$$

How can we simulate values T_x from the distribution F_x? Easy peasy. You know that the distribution function F_x apply to its random variable T_x will be uniformly distributed on $[0, 1]$. Proof: $\mathbb{P}[F_x(T_x) < u] = \mathbb{P}[T_x < F_x^{-1}(u)] = F_x^{-1}(F_x(u)) = u$, so $F_x(T_x)$ must be uniformly distributed. Therefore,

$F_x(T_x) \sim U[0,1]$

So,

$$T_x \sim F_x^{-1}(U[0,1]).$$

Generate random values $u_1, \ldots, u_n$, plug them into F_x^{-1} and you will obtain random samples for T_x. The challenge usually is that F_x^{-1} is not always easy to find explicitly (or even possible), so you will have to use a numerical method. Given u you want to find t_u such that

$$F_x(t_u) = u.$$

For that you can use the function *uniroot* or *optimize* in R.

Exercise 2.2 (Disability income insurance)

A disability income insurance provides benefits during periods of sickness. There are no benefits after recovery. Typically, such a policy provides an annuity while the person is sick, whereas premiums are paid while the insured is healthy. On the other hand, it could be also used for valuing lump sum payments in the case of sickness of death.

Assume in this model that the state of the insured $X_t \in S$ is described by a regular Markov chain with state space $S = \{*, \diamond, \dagger\}$ where $* =$"healthy", $\diamond =$"sick" and $\dagger =$"dead". Suppose that the transition rates are given by the Gompertz-Makeham model as follows

$$\mu_{*\diamond}(t) = a_1 + b_1 \exp(c_1 t) \quad \mu_{\diamond *}(t) = 0.1\mu_{*\diamond}(t), \quad \mu_{*\dagger}(t) = a_2 + b_2 \exp(c_2 t) \quad \mu_{\diamond\dagger}(t) = \mu_{*\dagger}(t),$$

where $a_1 = 4 \cdot 10^{-4}$, $b_1 = 3.4674 \cdot 10^{-6}$, $c_1 = 0.138155$, $a_2 = 5 \cdot 10^{-4}$, $b_2 = 7.5858 \cdot 10^{-5}$ and $c_2 = 0.087498$. Compute $p_{**}(x, x+10)$ and $p_{*\diamond}(x, x+10)$ for $x = 60$ (years). Simulate and draw the graphs of each transition probability for different values of x, say $x \in [0, 100]$.

Solution: Let $P(s,t)$, $0 \leqslant s \leqslant t$ denote the transition probability matrix and $\Lambda(t)$ the transition rate matrix. Then Kolmogorov's forward equation is given by

$$\frac{d}{dt}P(s,t) = P(s,t)\Lambda(t).$$

This can be written in component form as

$$\frac{d}{dt}p_{ij}(s,t) = \sum_{k \in S} p_{ik}(s,t)\mu_{kj}(t).$$

Separating the case $k = j$ we have

$$\frac{d}{dt}p_{ij}(s,t) = -p_{ij}(s,t)\mu_j(t) + \sum_{k \neq j} p_{ik}(s,t)\mu_{kj}(t).$$

In the case were recovery is allowed, the equation becomes more involved, as p_{*ast} and $p_{*\diamond}$ depend upon each other and the system of linear equations is interrelated. This means that, in order to solve the equation for finding $P(s,t)$ we need to compute the exponential matrix

$e^{\int_s^t \Lambda(u)du}$ which requires finding the Jordan decomposition of $\Lambda(t)$ (eigenvalues, eigenspaces, etc.) and then integrate. For this reason, it may be useful to integrate p_{ij} stepwise by using a numerical method. Also, because once we know how to implement a numerical method, we can consider more complex state spaces with several interrelated states. In the following we present two of the most well-known methods, the Euler method and the Runge-Kutta methods.

Euler scheme

Consider the ordinary differential equation (ODE) given by

$$\frac{d}{dt}x(t) = f(t, x(t)), \quad x(t_0) = x_0 \in \mathbb{R}, \quad t \leqslant 0.$$

Fix a small time step $h > 0$, initial time t_0 and initial condition $x_0 = x(t_0)$, then for each time step $t_n = t_0 + nh$, $n \geqslant 0$ (observe that $t_{n+1} = t_n + h$) we can find an approximation of the value of $x(t_{n+1})$ using the following recursive scheme

$$x_{n+1} = x_n + hf(t_n, x_n), \quad n \geqslant 0.$$

The value of x_n is an approximation of the solution to the ODE at time t_n: $x_n \approx x(t_n)$.

In the case where $f(t, x) = A(t)x$ (in matrix form too) then

$$x_{n+1} = x_n + hA(t_n)x_n, \quad n \geqslant 0.$$

This is the most primitive method and accumulated error is of order $O(h)$.

Runge-Kutta scheme

Consider the ordinary differential equation (ODE) given by

$$\frac{d}{dt}x(t) = f(t, x(t)), \quad x(t_0) = x_0 \in \mathbb{R}, \quad t \leqslant 0.$$

Fix a small time step $h > 0$, initial time t_0 and initial condition $x_0 = x(t_0)$, then for each time step $t_n = t_0 + nh$, $n \geqslant 0$ (observe that $t_{n+1} = t_n + h$) we can find an approximation of the value of $x(t_{n+1})$ using the following recursive scheme

$$x_{n+1} = x_n + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right), \quad n \geqslant 0,$$

where

$$\begin{aligned}
k_1 &= hf(t_n, x_n),\\
k_2 &= hf\left(t_n + \frac{h}{2}, x_n + \frac{k_1}{2}\right),\\
k_3 &= hf\left(t_n + \frac{h}{2}, x_n + \frac{k_2}{2}\right),\\
k_4 &= hf(t_n + h, x_n + k_3).
\end{aligned}$$

If f does not depend on the solution x, then the ODE is a regular integral and the Runge-Kutta method corresponds to Simpson's rule for numerical integration.

The global error is of order $O(h^4)$.

Our setting:

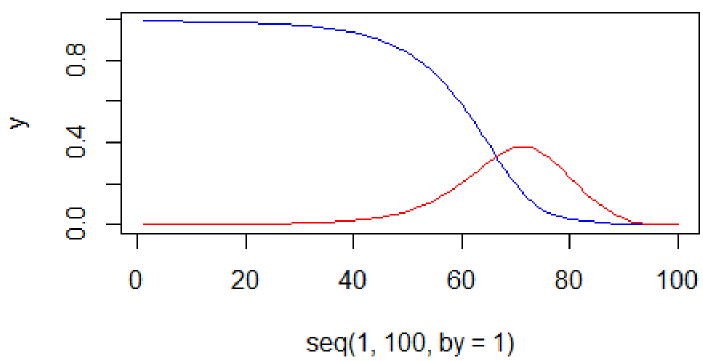
We have a two-dimensional setting, so $\vec{x} = (p_{**}(s,\cdot), p_{*\diamond}(s,\cdot))$. Then using an Euler scheme starting from $t_0 = s$ we have

$$p_{**}(s, s+(n+1)h) = p_{**}(s, s+nh) - \mu_*(s+nh)p_{**}(s, s+nh) + \mu_{\diamond *}(s+nh)p_{*\diamond}(s, s+nh)$$
$$p_{*\diamond}(s, s+(n+1)h) = p_{*\diamond}(s, s+nh) - \mu_{*\diamond}(s+nh)p_{**}(s, s+nh) - \mu_{\diamond}(s+nh)p_{*\diamond}(s, s+nh),$$

where

$$\mu_*(s+nh) = \mu_{*\diamond}(s+nh) + \mu_{*\dagger}(s+nh), \quad \mu_\diamond(s+nh) = \mu_{\diamond *}(s+nh) + \mu_{\diamond\dagger}(s+nh).$$

We used a Runge-Kutta integrator to plot $x \mapsto p_{**}(x, x+10)$ and $p_{*\diamond}(x, x+10)$ for $x \in [0, 100]$ and obtained



In particular, $p_{**}(60, 70) = 0.5859194$ and $p_{*\diamond}(60, 70) = 0.2030865$.

Exercise 2.3

Assume an endowment policy with 250 000$ death benefit and 125 000$ survival benefit. The yearly premium (in continuous time) is given by 2 200$. Suppose that the insured is 25 years at the beginning of the contract, which expires at the age of 65. Further, death occurs at 45 years. Let $\delta(s) = \log(1 + 0.03)$ (ca. 2.96%). Compute the prospective reserve $V^+(t, A)$ for $t = 30$ years.

<u>Solution:</u>

Let us first define the policy functions of this policy:

$$a_*(t) = \begin{cases} -2200(t-25), & t \in [25, 65), \\ -2200(65-25) + 125000, & \text{else.} \end{cases}, \quad a_{*\dagger}(t) = \begin{cases} 250000, & t \geqslant 65, \\ 0, & \text{else.} \end{cases}$$

The prospective reserve given the cash flow A is defined by

$$V^+(30, A) = \frac{1}{v(t,\omega)} \int_{[t,\infty)} v(s,\omega)\, dA(s,\omega).$$

Here, v is deterministic and ω is the event that the insured dies at the age of 45. What is the prospective reserve at the age of 30?

$$\begin{aligned} V^+(30, A) &= \frac{1}{v(30)} \int_{[30,\infty)} v(s) dA(s, \omega) \\ &= \frac{1}{v(30)} \int_{[30,\infty)} v(s) dA_*(s, \omega) + \frac{1}{v(30)} \int_{[30,\infty)} v(s) dA_{*\dagger}(s, \omega). \end{aligned}$$

Denote by $\dot{a}_*(t)$ a function such that $da_*(t) = \dot{a}_*(t)dt$ a.e. In this case, $\dot{a}_*(t) = -2200$ if $t \in [25, 65)$. On the one hand,

$$\int_{[30,\infty)} v(s) dA_*(s, \omega) = \int_{[30,\infty)} v(s) \mathbf{1}_{\{X_s(\omega)=*\}} da_*(s) = \int_{[30,45)} v(s) \dot{a}_*(s) ds = \frac{2200}{\delta} \left(e^{-45\delta} - e^{-30\delta}\right).$$

On the other hand,

$$\begin{aligned} \int_{[30,\infty)} v(s) dA_{*\dagger}(s, \omega) &= \int_{[30,\infty)} v(s) a_{*\dagger}(s) dN_{*\dagger}(s, \omega) \\ &= \sum_{\substack{s>30 \\ *\rightsquigarrow\dagger}} v(s) a_{*\dagger}(s) \mathbf{1}_{\{X_{s-}(\omega)=*, X_s(\omega)=\dagger\}} \\ &= v(45) a_{*\dagger}(45) \\ & 250000 e^{-45\delta}. \end{aligned}$$

Altogether,

$$V^+(30, A) = e^{30\delta} \frac{2200}{\delta} \left(e^{-45\delta} - e^{-30\delta}\right) + e^{30\delta} 250000 e^{-45\delta} = 133810\$$$