STK4500: Life Insurance and Finance

Exercise list 3: Solutions

Exercise 3.1 (Interest rates in a periodic economy)

Consider interest rates in a "periodic economy" with a period of 8 states. Let us assume that such interest rates are modeled by a homogeneous Markov chain i_t, $t = 0, 1, 2, \ldots$, with the following transition probabilities for the corresponding states:

State	Interest rate (%)	p_{ii}	p_{ii+1}	p_{ii+2}	
0	Start	5.0	0.1	0.7	0.2
1	Increasing interest	5.5	0.1	0.7	0.2
2	Max. interest	6.0	0.1	0.7	0.2
3	Decreasing interest	5.5	0.1	0.7	0.2
4	Average	5.0	0.1	0.7	0.2
5	Decreasing interest	4.5	0.1	0.7	0.2
6	Min. interest	4.0	0.1	0.7	0.2
7	Increasing interest	4.5	0.1	0.7	0.2

Calculate the probability that i_t is in state 2, given that i_0 is in state zero for $t = 4$.

Solution:

Chapman-Kolmogorov equation and the fact that this chain is time-homogeneous implies that

$$P(0,4) = P(0,1)P(1,2)P(2,3)P(3,4) = P^4$$

Hence, denote by $e_i = (0, \ldots, 1, \ldots, 0)^t$ where 1 is in the i-th position, then

$$p_{0,2}^{(4)} := P(i_4 = 2 | i_0 = 0) = e_1^t P^4 e_3 = 0.0302$$

Exercise 3.2

Consider a disability insurance. Suppose for this model that there is no waiting period and that the disability pension is given by the fixed amount of 20 000\$ per year until the age of 65 years. Assume that annual premiums are 2 500\$ until the age of 65 years and that the age of the insured at the beginning of the contract is $x_0 = 30$ years.

Determine the policy functions $a_i(t)$ (generalized pension payments) and $a_{ij}(t)$ (generalized capital benefits).

Solution:

Continuous time:

The policy functions that define this insurance are

$$a_*(t) = \begin{cases} 0, & t < 0, \\ -2\,500 \cdot t, & t \in [0,35), \\ -2\,500 \cdot 35, & t \geqslant 35. \end{cases} \quad , \quad a_\diamond(t) = \begin{cases} 0, & t < 0, \\ 20\,000 \cdot t, & t \in [0,35), \\ 20\,000 \cdot 35, & t \geqslant 35. \end{cases}$$

There are no functions $a_{*\dagger}$ because there is no death benefit (benefit from changing from $*$ to $\dagger$), there is no $a_{*\diamond}(t)$ either because there is no benefit or penalty from changing from $*$ to $\diamond$. The only contributions are the premiums (to the insurer, that is why there is a minus) and the disability pension.

Discrete time:

Although it is not asked, it is worth noting how the policy functions would look like in discrete time:

$$a_*^{Pre}(n) = \begin{cases} 0, & n < 0, \\ -2\,500, & n \in \{0,\dots,34\}, \end{cases} \qquad a_\diamond^{Pre}(n) = \begin{cases} 0, & n < 0, \\ 20\,000, & n \in \{0,\dots,34\}. \end{cases}$$

Exercise 3.3

Let us assume in Exercise 3.2 a permanent disability model with transition rates given by

$$\mu_{*\diamond}(t) = 0.0279, \quad \mu_{*\dagger}(t) = 0.0229, \quad \mu_{\diamond\dagger}(t) = \mu_{*\dagger}(t).$$

Further, let the (constant) force of interest r_t be 3%. Calculate the total reserve $V_j(t,A)$ for all states j when the insured is 60 years old, i.e. $t = 30$. Calculate the total reserve $V_j(t,A)$ also when reactivation is allowed assuming $\mu_{\diamond *}(t) = 0.1\mu_{*\diamond}(t)$.

Solution:

The (forward) equations for p_{**} and $p_{\diamond *}$ are

$$\frac{d}{dt}p_{**}(s,t) = p_{**}(s,t)\mu_{**}(t) + p_{*\diamond}(s,t)\mu_{\diamond *}(t),$$
$$\frac{d}{dt}p_{*\diamond}(s,t) = p_{**}(s,t)\mu_{*\diamond}(t) + p_{*\diamond}(s,t)\mu_{\diamond\diamond}(t).$$

Using an Euler scheme we find the transition probabilites. The following is an illustration of the result for $s = 0$.

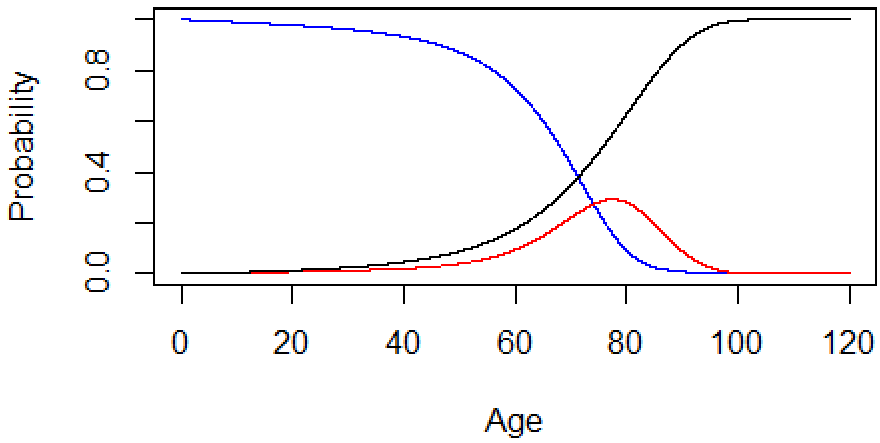

FIGURE 1: Probabilities $p_{**}(0,\cdot), p_{*\diamond}(0,\cdot), p_{*\dagger}(0,\cdot)$.

The explicit formula for the prospective reserve is

$$V_j^+(t,A) = \frac{1}{v(t)}\left[\sum_{k\in S}\int_t^\infty v(s)p_{jk}^x(t,s)da_k(s) + \sum_{k\in S}\int_t^\infty v(s)p_{jk}^x(t,s)\sum_{l\neq k} a_{kl}(s)\mu_{kl}^x(s)ds\right].$$

We have $\dot{a}_*(t) = -2\,500$, $\dot{a}_\diamond(t) = 20\,000$. In our setting, $x = 30$, $t = 30$, $T = 35$ and the remaining contract interval is $[30, 35]$, hence

$$V_*^+(30,A) = \frac{1}{v(30)}\left[\int_{30}^{35} v(s)p_{**}(60,s)(-2\,500)ds + \int_{30}^{35} v(s)p_{*\diamond}(60,s)20\,000ds\right] = -5\,193.92\$$$

Since $p_{\diamond *}(s,t) = 0$ we also get

$$V_\diamond^+(30,A) = \frac{1}{v(30)}\int_{30}^{35} v(s)\overline{p}_{\diamond\diamond}(60,s)ds20\,000 = 87\,962.7\$$$