STK4500: Life Insurance and Finance

Exercise list 4: Solutions

Exercise 4.1 (Term insurance in discrete time)

A 10 years old policy is issued to a life aged 50. The sum insured, payable at the end of the year of death, is 200 000$ and premiums π are paid yearly throughout the term.

Assume that the intensity rate $r = 2.5\%$ and that

$$\mu_{*\dagger}(t) = 0.002 + 0.0005(t - 50).$$

Use the equivalence principle to calculate the annual premium π and the prospective reserves $V_*^+(t)$, $t = 0, \ldots, 10$.

Solution:

First, we will need the transition probabilities, since we only have one possible transition, then

$$p_{**}(s,t) = \overline{p}_{**}(s,t) = e^{-\int_s^t \mu_{*\dagger}(u)du} = e^{0.023(t-s) - \frac{0.0005}{2}(t^2 - s^2)}.$$

In particular,

$$p_{**}(t, t+1) = e^{0.02325 - 0.0005t}.$$

Information about this policy: $x = 50$, $T = 10$, death benefit: 200000\$, $r = 2.5\%$.

The policy functions are

$$a_*^{Pre}(t) = \begin{cases} -\pi, \text{ if } t \in \{0, \ldots, 9\} \\ 0 \text{ else} \end{cases} \qquad a_{*\dagger}^{Post}(t) = \begin{cases} 200\,000, \text{ if } t \in \{0, \ldots, 9\} \\ 0 \text{ else} \end{cases}.$$

Recall $A(t) = A_*(t) + A_{*\dagger}(t) = \sum_{s\in[0,t]} \mathbf{1}_{\{X_s = *\}} a_*^{Pre}(\tau) + \sum_{\substack{s\in[0,t] \\ i\rightsquigarrow j}} a_{*\dagger}^{Post}(s)$.

The reserve at time t given that the insured is alive at time t is given by

$$V_*^+(t, A) = \frac{1}{v(t)} \sum_{\tau \geqslant t} v(\tau) p_{**}^x(t,\tau) a_*^{Pre}(\tau) + \frac{1}{v(t)} \sum_{\tau \geqslant t} v(\tau+1) p_{**}^x(t,\tau) p_{*\dagger}^x(\tau, \tau+1) a_{*\dagger}^{Post}(\tau).$$

Further,

$$V_*^+(t, A) = \frac{1}{v(t)} \left[-\pi \sum_{\tau=t}^{9} v(\tau) p_{**}^x(t,\tau) + 200\,000 \sum_{\tau=t}^{9} v(\tau+1) p_{**}^x(t,\tau) p_{*\dagger}^x(\tau, \tau+1) \right].$$

The equivalence principle states that the fair premium π_0 should be such that $V_*^+(0) = 0$. Hence,

$$\pi_0 = \frac{200\,000 \sum_{\tau=0}^{9} v(\tau+1) p_{**}^x(0,\tau) p_{*\dagger}^x(\tau,\tau+1)}{\sum_{\tau=0}^{9} v(\tau) p_{**}^x(0,\tau)} = 852.25\$$$

Alternative: Using Thiele's difference equation.

We know that

$$V_*^+(t) = V_*(t, A_*) + V_*(t, A_{*\dagger}),$$

where the first term corresponds to the reserves for the premiums and the last for benefits. Moreover, observe that

$$V_*^+(t) = V_*(t, A_*) + V_*(t, A_{*\dagger}) = \pi V_*(t, \widetilde{A}_*) + V_*(t, A_{*\dagger}),$$

where $\widetilde{A}_*$ is the payout function associated to the policy function paying 1 monetary unit a year:

$$\widetilde{a}_*^{Pre}(t) = \begin{cases} -1, & \text{if } t \in \{0, \dots, 9\} \\ 0 \text{ else} \end{cases} .$$

We then use Thiele's difference equation to compute $V_*(t, \widetilde{A}_*)$ and $V_*(t, A_{*\dagger})$ separately.

First, to compute $V_*(t, \widetilde{A}_*)$, we use Thiele's equation assuming $a_{*\dagger}^{Post} = 0$ and hence

$$V_*(t, \widetilde{A}_*) = -1 + e^{-\delta} p_{**}^x(t, t+1) V_*(t+1, \widetilde{A}_*), \quad V_*(10, \widetilde{A}_*) = 0.$$

Secondly,

$$V_*(t, A_{*\dagger}) = e^{-\delta} p_{**}^x(t, t+1) V_*(t+1, A_{*\dagger}) + e^{-\delta} p_{*\dagger}^x(t, t+1) 200\,000, \quad V_{*\dagger}(10, A_*) = 0.$$

Using R we obtain

$$V_*(0, \widetilde{A}_*) = -8.824066, \quad V_*(0, A_{*\dagger}) = 7\,520.29.$$

Then the premium is such that $V_*^+(0, A) = 0$, i.e.

$$\pi = -\frac{V_*(0, A_{*\dagger})}{V_*(0, \widetilde{A}_*)} = 7520.29/8.824066 = 852.25\$$$

Exercise 4.2 (Permanent disability insurance in discrete time)

Consider a permanent disability insurance. Let $x = 45$ years be the initial age of the insured and let 65 be the age at maturity. Further assume that $r = 3\%$ (intensity rate) and that the yearly disability pension is given by 12 000\$.

Compute the reserves of the disability pension for each of the following cases (see List 1, Exercise 5):

(i)

$$\mu_{*\diamond}(t) = 0.0279 \quad \mu_{*\dagger}(t) = 0.0229 \quad \mu_{\diamond\dagger}(t) = \mu_{*\dagger}(t).$$

(ii) Gompertz-Makeham model as follows

$$\mu_{*\diamond}(t) = a_1 + b_1 \exp(c_1 t) \quad \mu_{*\dagger}(t) = a_2 + b_2 \exp(c_2 t) \quad \mu_{\diamond\dagger}(t) = \mu_{*\dagger}(t),$$

where $a_1 = 4 \cdot 10^{-4}$, $b_1 = 3.4674 \cdot 10^{-6}$, $c_1 = 0.138155$, $a_2 = 5 \cdot 10^{-4}$, $b_2 = 7.5858 \cdot 10^{-5}$ and $c_2 = 0.087498$.

Solution:

We use Euler scheme to compute the transition probabilities that we already programmed for List 3 (Problem 3).

The policy function for this insurance is

$$a_\diamond^{Pre}(t) = \begin{cases} 12\,000 \text{ if } t \in \{0, \dots, 19\} \\ 0 \text{ else} \end{cases} .$$

From Thiele's difference equation it is clear that $V_\dagger^+(t) = 0$ for all t and hence

$$\begin{aligned} V_*^+(t) &= e^{-\delta} p_{**}^x(t, t+1) V_*^+(t+1) + e^{-r} p_{*\diamond}^x(t, t+1) V_\diamond^+(t+1), \\ V_\diamond^+(t) &= a_\diamond^{Pre}(t) + e^{-r} p_{\diamond\diamond}^x(t, t+1) V_\diamond^+(t+1), \end{aligned}$$

the terminal conditions are $V_*^+(20) = V_\diamond^+(20) = 0$.

Using this recursion we obtain

$$V_*^+(0, A) = 7\,248.11, \quad V_\diamond^+(0, A) = 175\,159.5$$