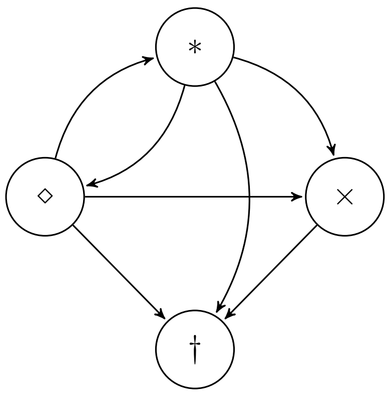
STK4500: Life Insurance and Finance

Exercise list 5: Solutions

Exercise 5.1

Let us consider an insurance policy model which combines benefits of a disability income insurance (see Exercise list 2, Exercise 2) and benefits in the case of critical illness. The state space for the driving Markov chain X is $S = \{*, \diamond, \times, \dagger\}$ where $*$ means healthy, $\diamond$ means sick, $\times$ means critically ill (with no possibility of recovery) and $\dagger$ is death, i.e.



The transition intensities are given by

$$\begin{aligned}
&\mu_{*\diamond}(t) = a_1 + b_1 \exp(c_1 t), \quad \mu_{*\times}(t) = 0.05\mu_{*\diamond}(t), \quad \mu_{*\dagger}(t) = a_2 + b_2 \exp(c_2 t),\\
&\mu_{\diamond *}(t) = 0.1\mu_{*\diamond}(t), \quad \mu_{\diamond\times}(t) = \mu_{*\times}(t), \quad \mu_{\diamond\dagger}(t) = \mu_{*\dagger}(t),\\
&\mu_{\times\dagger}(t) = 1.2\mu_{*\dagger}(t).
\end{aligned}$$

where $a_1 = 4 \cdot 10^{-4}$, $b_1 = 3.4674 \cdot 10^{-6}$, $c_1 = 0.138155$, $a_2 = 5 \cdot 10^{-4}$, $b_2 = 7.5858 \cdot 10^{-5}$ and $c_2 = 0.087498$. Use the Euler approximation scheme with step size $h = \frac{1}{12}$ (1 month) to compute $p_{**}(x, x+35)$ for an insured aged $x = 30$ years. You may also plot $t \mapsto p_{**}(30, t)$, $t \geqslant 30$.

Solution:

The transition probability matrix $P(s,t) = \{p_{ij}(s,t)\}_{i,j \in S}$ and rate matrix $\Lambda(t) = \{\mu_{ij}\}_{i,j\in S}$ look like this

$$P(s,t) = \begin{pmatrix} p_{**} & p_{*\diamond} & p_{*\times} & p_{*\dagger} \\ p_{\diamond *} & p_{\diamond\diamond} & p_{\diamond\times} & p_{\diamond\dagger} \\ 0 & 0 & p_{\times\times} & p_{\times\dagger} \\ 0 & 0 & 0 & 1 \end{pmatrix}(s,t), \quad \Lambda(t) = \begin{pmatrix} \mu_{**} & \mu_{*\diamond} & \mu_{*\times} & \mu_{*\dagger} \\ \mu_{\diamond *} & \mu_{\diamond\diamond} & \mu_{\diamond\times} & \mu_{\diamond\dagger} \\ 0 & 0 & -\mu_{\times\dagger} & \mu_{\times\dagger} \\ 0 & 0 & 0 & 0 \end{pmatrix}(t).$$

Kolmogorov's forward equation in matrix form is

$$\frac{d}{dt}P(s,t) = P(s,t)\Lambda(t), \quad P(s,s) = Id.$$

Let $s > 0$, $t < s$ and $h > 0$ be fixed, define $N = \frac{t-s}{h}$ the number of iterations. Then the Euler scheme is defined for each $t_n = s + nh$, $n = 0, \ldots, N$ as follows

$$P(s, t_{n+1}) = P(s, t_n) + hP(s, t_n)\Lambda(t_n), \quad n \geqslant 0,$$

where $P(s, t_0) = P(s, s) = Id$. The matrix $P(s, t_N)$ approximates the values of the matrix $P(s, t)$. We have implemented this in R and obtained the following plots for $s = 0$ and $t = 100$, and $s = 30$ to $t = 100$.

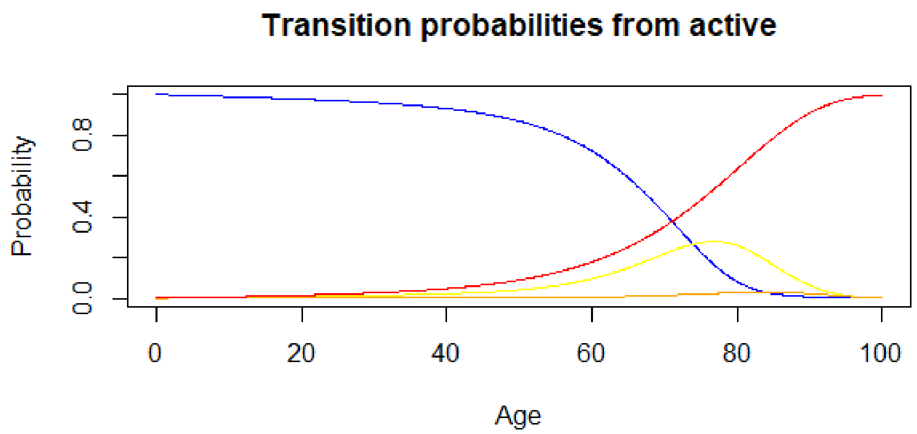


FIGURE 1: Transition probabilities starting from $s = 0$ in the active state up to $t = 100$. The blue color is $p_{**}(0, \cdot)$ and the darker colours correspond to the states $\diamond$, $\times$ and $\dagger$, respectively.

For $t \mapsto p_{*,j}(30, t)$ we have

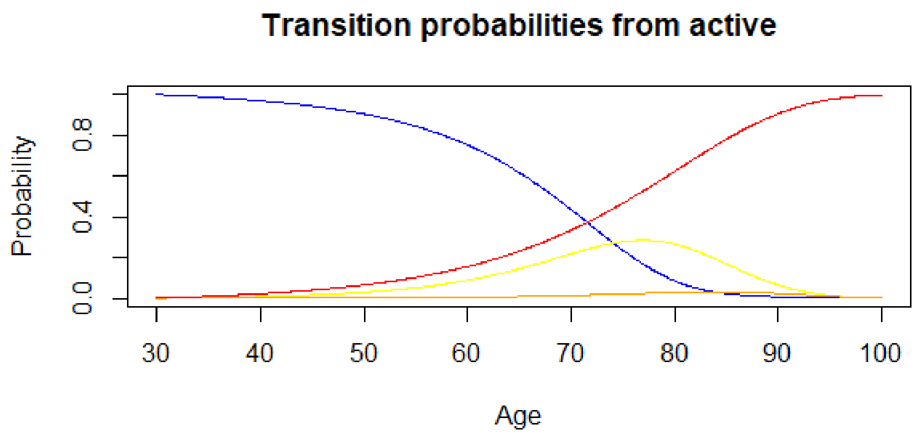


FIGURE 2: Transition probabilities starting from $s = 30$ in the active state up to $t = 100$. The blue color is $p_{**}(30, \cdot)$ and the darker colours correspond to the states $\diamond$, $\times$ and $\dagger$, respectively.

The transition matrix $P(30,65)$ was found to be

$$P(30,65) = \begin{pmatrix} 0.61908042 & 0.1435257 & 0.007907848 & 0.2294861 \\ 0.01435257 & 0.7482535 & 0.007907848 & 0.2294861 \\ 0.00000000 & 0.0000000 & 0.731534516 & 0.2684655 \\ 0.00000000 & 0.0000000 & 0.000000000 & 1.0000000 \end{pmatrix}$$

Hence, $p_{**}(30,65) = 0.61908042$. If you wish to do this by hand, observe that p_{**} does only depend on the transient class $\{*, \diamond\}$ since all other states never return to $\{*, \diamond\}$. Thus you will only obtain a two dimensional SDE for p_{**} and $p_{*\diamond}$ and you can carry out the Euler scheme iterating two linear ODEs.

Exercise 5.2

Consider a 10-years disability income insurance to a healthy life aged 60 years. Payments of 20 000\$ are provided by the insurer (continuously in time), while the insured is in the disabled state. A death benefit of 50 000\$ is immediately payable on death. Further, it is also required that premiums are payable continuously while the insured is in the healthy state. Assume that $r = 5\%$ (intensity rate) and that other expenses are ignored. The transition rates are given by

$$\mu_{*\diamond}(t) = a_1 + b_1 \exp(c_1 t) \quad \mu_{\diamond *}(t) = 0.1\mu_{*\diamond}(t), \quad \mu_{*\dagger}(t) = a_2 + b_2 \exp(c_2 t) \quad \mu_{\diamond\dagger}(t) = \mu_{*\dagger}(t),$$

where $a_1 = 4 \cdot 10^{-4}$, $b_1 = 3.4674 \cdot 10^{-6}$, $c_1 = 0.138155$, $a_2 = 5 \cdot 10^{-4}$, $b_2 = 7.5858 \cdot 10^{-5}$ and $c_2 = 0.087498$.

Calculate the (constant) annual premiums π for this policy.

Solution:

There are two ways of solving this problem: 1) Thiele's differential equation or 2) Using the explicit formula for $V_*^+(t, A)$.

The policy functions are given by

$$a_*(t) = \begin{cases} -\pi t, \text{ if } t \in [0,10) \\ -\pi 10, \text{ if } t \geqslant 10 \\ 0, \text{ else} \end{cases} \qquad a_\diamond(t) = \begin{cases} -20\,000t, \text{ if } t \in [0,10) \\ -20\,000 \cdot 10, \text{ if } t \geqslant 10 \\ 0, \text{ else} \end{cases}$$

and for transitions

$$a_{*\dagger}(t) = \begin{cases} 50\,000, \text{ if } t \in [0,10) \\ 0, \text{ else} \end{cases} \qquad a_{\diamond\dagger}(t) = \begin{cases} 50\,000, \text{ if } t \in [0,10) \\ 0, \text{ else} \end{cases}.$$

We know that the explicit formula is

$$V_*^+(t, A) = \frac{1}{v(t)} \Bigg[\int_t^\infty v(s) p_{**}^x(t,s) da_*(s) + \int_t^\infty v(s) p_{*\diamond}^x(t,s) da_\diamond(s) + \int_t^\infty v(s) p_{**}^x(t,s) \mu_{*\dagger}(s) a_{*\dagger}(s) ds + \int_t^\infty v(s) p_{*\diamond}^x(t,s) \mu_{\diamond\dagger}^x(s) a_{\diamond\dagger}(s) ds \Bigg].$$

Observe that the Riemann-Stiltjes integrals have continuous integrators so we do not need to add any jumps. We have $\dot{a}_*(t) = -\pi$ and $\dot{a}_\diamond(t) = 50\,000$, hence for $t \in [0, 10]$ we have

$$V_*^+(t, A) = \frac{1}{v(t)}\Bigg[-\pi \int_t^{10} v(s)p_{**}^x(t, s)ds + 20\,000 \int_t^{10} v(s)p_{*\diamond}^x(t, s)ds + 50\,000 \int_t^{10} v(s)p_{**}^x(t, s)\mu_{*\dagger}^x(s)ds + 50\,000 \int_t^{10} v(s)p_{*\diamond}^x(t, s)\mu_{\diamond\dagger}^x(s)ds\Bigg].$$

Using the equivalence principle, i.e. $V_*^+(0, A) = 0$ we obtain

$$\pi = \frac{50\,000 \int_0^{10} v(s)p_{**}(x, x+s)\mu_{*\dagger}(x+s)ds + 50\,000 \int_0^{10} v(s)p_{*\diamond}(x, x+s)\mu_{\diamond\dagger}(x+s)ds + 20\,000 \int_0^{10} v(s)p_{*\diamond}(x, x+s)ds}{\int_0^{10} v(s)p_{**}(x, x+s)ds} \approx 3260.25\$$$

where $x = 60$, $r = 0.05$ and we used R to complete the last computation. The integrals were computed using a rudimentary Riemann sum, i.e.

$$\int_a^b f(s)ds \approx \sum_{i=1}^{(b-a)/h} f(a + ih)h,$$

where a, b and h are such that $(b - a)/h$ is integer. In our case $a = 0$, $b = 10$ and $h = 1/12$. Probably, Newton-Cotes formulae, e.g. Simpson's rule would lead to a better estimate since the integrands are always positive.