STK4500: Life Insurance and Finance

Exercise list 6: Solutions

Exercise 6.1

An endowment policy is issued to a life aged $x = 40$ with 60 years age maturity. The death benefit is assumed to be 220 000\$, whereas the endowment amount is given by 115 000\$. Further, let the constant technical interest rate r be 3% and the mortality rates be given by

$$\mu_{*\dagger}(x+t) = 0.0018 + 0.0004t, \quad t \geqslant 0.$$

Calculate the second moment of the present value of the benefit payments V_t^+, given $X_t = *$ for $t = 10, \dots, 20$.

Solution:

Define the second order moment $V_2(t) := E[(V_t^+)^2 | X_t = *]$. Then Thiele's difference equation for the second order moment (see Theorem 5.2 in the lecture notes) implies

$$\begin{aligned} V_2(t) &= e^{-2r} p_{**}^x(t,t+1)((a_{**}^{Post}(t))^2 + 2a_{**}^{Post}(t)V_*^+(t+1) + E[(V_{t+1}^+)^2|X_{t+1} = *]) \\ &+ e^{-2r} p_{*\dagger}^x(t,t+1)\left((a_{*\dagger}^{Post}(t))^2 + 2a_{*\dagger}^{Post}(t)\underbrace{V_\dagger^+(t+1))}_{=0} + \underbrace{E[(V_{t+1}^+)^2|X_{t+1} = \dagger]}_{=0}\right) \end{aligned}$$

and

$$V_*^+(t) = e^{-r} p_{**}^x(t,t+1)\left(a_{**}^{Post}(t) + V_*^+(t+1)\right) + e^{-r} p_{*\dagger}^x(t,t+1)\left(a_{*\dagger}^{Post}(t) + \underbrace{V_\dagger^+(t+1)}_{=0}\right),$$

where

$$p_{**}^x(t,t+1) = \exp\left(-\int_t^{t+1} \mu_{*\dagger}(x+s)ds\right)$$

and

$$p_{*\dagger}^x(t,t+1) = 1 - p_{**}^x(t,t+1).$$

Here, $r = 0.03$. Then

$$\begin{aligned} &V_2(20) = (115000)^2, \quad V_2(19) = 3.7443 \cdot 10^10, \quad V_2(18) = 4.17426 \cdot 10^8 \\ &V_2(17) = 3.99357 \cdot 10^8, \quad V_2(16) = 3.8128 \cdot 10^8, \quad V_2(15) = 3.63197 \cdot 10^8, \\ &V_2(14) = 3.45106 \cdot 10^8, \quad V_2(13) = 3.27007 \cdot 10^8, \quad V_2(12) = 3.08902 \cdot 10^8, \\ &V_2(11) = 2.90789 \cdot 10^8, \quad V_2(10) = 2.7267 \cdot 10^8. \end{aligned}$$

Exercise 6.2

Consider a 10-years permanent disability insurance (in discrete time) issued to a healthy life aged 60. Yearly payments of 30 000$ are made while the insured is disabled. A death benefit of 50 000$ is paid at the end of the year of death. Assume that $r = 5\%$ and that the transition rates are given as in Exercise list 1, Exercise 5 (i), that is

$$\mu_{*\diamond}(t) = 0.0279 \quad \mu_{*\dagger}(t) = 0.0229 \quad \mu_{\diamond\dagger}(t) = \mu_{*\dagger}(t).$$

Compute $\mathcal{P}[V_t^+ < u | X_t = \diamond]$ for $t = 7$ years and $u = 60\,000$\$.
Solution:
The policy functions (without premium payments) are given by

$$a_{\diamond}^{Pre}(t) = \begin{cases} 20\,000, & \text{if } t = 0, \dots, 9, \\ 0, & \text{else.} \end{cases} \quad , \quad a_{*\dagger}^{Post}(t) = a_{*\diamond}^{Post}(t) = \begin{cases} 50\,000, & \text{if } t = 0, \dots, 9, \\ 0, & \text{else.} \end{cases}$$

Theorem 5.7 from the lecture notes gives a recursion formula for $P_*(t,u) := P(V_t^+ < u | X_t = *)$:

$$\begin{aligned} P(t,u) := & \, P(V_t^+ < u | X_t = *) \\ & + p_{**}^x(t,t+1) P_*(t+1, e^r u) + p_{*\diamond}^x(t,t+1) P_{\diamond}(t+1, e^r u) \\ & + p_{*\dagger}^x(t,t+1) P_{\dagger}(t+1, e^r u - a_{*\dagger}^{Post}(t)) \end{aligned}$$

We have

$$P_{\dagger}(t+1, e^r u - a_{*\dagger}^{Post}(t)) = \begin{cases} 1, & \text{if } e^r u - a_{*\dagger}^{Post}(t) \geqslant 0, \\ 0 & \text{else} \end{cases}$$

and

$$P_{\diamond}(t,u) = p_{\diamond\diamond}^x(t,t+1) P_{\diamond}(t+1, e^r(u - \underbrace{a_{\diamond}^{Pre}(t)}_{=20\,000})) + p_{\diamond\dagger}^x(t,t+1) P_{\dagger}(t+1, e^r(u - a_{\diamond}^{Pre}(t)) - a_{\diamond\dagger}^{Post}(t))$$

and

$$P_{\dagger}(t+1, e^r(u - a_{\diamond}^{Pre}(t) = \begin{cases} 1, & \text{if } e^r(u - a_{\diamond}^{Pre}(t)) - a_{\diamond\dagger}^{Post}(t) \geqslant 0, \\ 0 & \text{else} \end{cases} .$$

Then using that $V_{10}^+ = 0$ and the probabilities from Exercise 5 List 1 we get

$$P_{\diamond}(7, 60\,000) = P(V_7^+ < 60\,000 | X_7 = \diamond) = 0.9336$$

Exercise 6.3 (Mathematical reserve refund guarantee)

Consider a pension with refund guarantee issued to a life aged $x = 50$ years. Here the maturity of the pension is at the age of 60 and the refund guarantee is the death benefit given by the last mathematical reserve, which is paid at the end of the year of death. Further, the pension payments are 15 000\$ per year. Assume that $r = 3\%$ and that

$$\mu_{*\dagger}(x+t) = 0.002 + 0.0005, \quad t \geqslant 0.$$

Determine the prospective reserve of the pension and benefit payments (single premium) at the beginning of the contract.

Solution:

Let P denote the yearly pension. The policy functions are

$$a_*^{Pre}(t) = \begin{cases} P, & \text{if } t = 0,\dots,9, \\ 0, & \text{else.} \end{cases} \quad , \quad a_{*\dagger}^{Post}(t) = \begin{cases} e^r V_*^+(t), & \text{if } t = 0,\dots,9, \\ 0, & \text{else.} \end{cases}$$

Thiele's difference equation gives

$$\begin{aligned} V_*^+(t) &= a_*^{Pre}(t) + e^{-r} p_{**}^x(t,t+1) V_*^+(t+1) + e^{-r} p_{*\dagger}^x(t,t+1)\left(a_{*\dagger}^{Post} + V_\dagger^+(t+1)\right) \\ &= P + e^{-r} p_{**}^x(t,t+1) V_*^+(t+1) + p_{*\dagger}^x(t,t+1) V_*^+(t) \end{aligned}$$

for $t = 0,\dots,9$. As a result,

$$V_*^+(t) = \frac{1}{1 - p_{**}^x(t,t+1)}\left(P + e^{-r} p_{**}^x(t,t+1) V_*^+(t+1)\right)$$

where $P = 15\,000$ and the terminal condition is $V_*^+(10) = 0$. Hence,

$$V_*^+(9) = 15\,101.6\$, \quad V_*^+(8) = 29\,749.3\$, \quad \dots, \quad V_*^+(0) = 132\,122\$$$

Exercise 6.4

Consider the disability income insurances of Exercise list 5, Exercise 1. Assume here that the insurance company issues a policy to a life aged $x = 30$ with maturity 35 years. A lump sum payment of 100 000\$ is immediately made in the case of critical illness. In addition, a death benefit of 100 000\$ is instantly paid, provided that the insured has not already been paid a critical illness benefit. A yearly disability pension of 75 000\$ is payable while the life is disabled. Further, yearly premiums are payable continuously, if the insured is healthy. Other expenses are ignored and let $r = 5\%$.

Calculate the yearly premiums for this policy.

Solution: The policy functions are

$$a_*(t) = \begin{cases} -\pi t, & \text{if } t \in [0,5), \\ -\pi \cdot 5, & \text{if } t \geqslant 5 \end{cases} \qquad a_\diamond(t) = \begin{cases} 75\,000t, & \text{if } t \in [0,5), \\ 75\,000 \cdot 5, & \text{if } t \geqslant 5 \end{cases}$$

$$a_{*\dagger}(t) = a_{\diamond\dagger}(t) = \begin{cases} 100\,000, & \text{if } t \in [0,5), \\ 0, & \text{if } t \geqslant 5 \end{cases} \qquad a_{*\times}(t) = a_{\diamond\times}(t) = \begin{cases} 100\,000, & \text{if } t \in [0,5), \\ 0, & \text{if } t \geqslant 5 \end{cases}$$

and all other possibilities are 0.

The prospective reserve at initial time, that is when the insured is 30 (i.e. $t = 0$) is

$$V_*^+(0, A) = A_*^+(0, \widetilde{A}_*)\pi + A_*^+(0, \widetilde{A}),$$

where $A_*^+(0, \widetilde{A}_*)$ is the prospective reserve for premium payments associated to a policy function $\widetilde{a}_*$ paying 1 \$ yearly, (i.e. with premium equals to 1 and therefore we multiply outside by π), and $A_*^+(0,\widetilde{A})$ corresponds to the rest of the reserves for the pension and benefit payments (i.e. disability pension and death benefits).

From the explicit formula we know that

$$A_*^+(0,\widetilde{A}_*) = \int_0^5 v(s)p_{**}(x,x+s)d\widetilde{a}_*(s) = -\int_0^5 v(s)p_{**}(x,x+s)ds,$$

where we used the fact that $\widetilde{a}_*$ is continuous on $[0,5]$. Using numerical integration and the probabilities from Exercise 1 from List 5 we obtain

$$A_*^+(0,\widetilde{A}_*) = -4.471199$$

Finally, for $A_*^+(0,\widetilde{A})$ we have

$$\begin{aligned} A_*^+(0,\widetilde{A}) = \bigg[& \int_0^5 v(s)p_{*\diamond}(x,x+s)da_\diamond(s) \\ & + \int_0^5 v(s)p_{**}(x,x+s)\left(\mu_{*\times}(x+s)a_{*\times}(s) + \mu_{*\dagger}(x+s)a_{*\dagger}(s)\right)ds \\ & + \int_0^5 v(s)p_{*\diamond}(x,x+s)\left(\mu_{\diamond\times}(x+s)a_{\diamond\times}(s) + \mu_{\diamond\dagger}(x+s)a_{\diamond\dagger}(s)\right)ds \bigg]. \end{aligned}$$

Substituting

$$\begin{aligned} A_*^+(0,\widetilde{A}) = & \bigg[75\,000 \int_0^5 v(s)p_{*\diamond}(x,x+s)ds \\ & + 100\,000 \int_0^5 v(s)p_{**}(x,x+s)\left(\mu_{*\times}(x+s) + \mu_{*\dagger}(x+s)\right)ds \\ & + 100\,000 \int_0^5 v(s)p_{*\diamond}(x,x+s)\left(\mu_{\diamond\times}(x+s) + \mu_{\diamond\dagger}(x+s)\right)ds \bigg] \\ = & \ 1\,384.81\$. \end{aligned}$$

The yearly premium using the equivalence principle is

$$\pi = -\frac{A_*^+(0,\widetilde{A})}{A_*^+(0,\widetilde{A}_*)} = 309.72\$$$

This insurance is rather cheap given the fact that the insured is quite young and the probabilities of transitioning were very low.