STK4500: Life Insurance and Finance

Exercise list 7

Exercise 7.1

Let $f : [0, T] \to \mathbb{R}$ be a continuous function and $B = \{B_t, t \in [0, T]\}$ a Brownian motion. Show that

$$\int_0^T f(s) dB_s$$

is normally distributed with mean zero and variance given by

$$\int_0^T f(s)^2 ds.$$

Exercise 7.2

Let $B = \{B_t, t \in [0, T]\}$ be a Brownian motion.

(i) Compute $[B, B]_t$ using the definition of quadratic variation.

(ii) Use (i) to evaluate

$$\int_0^T B_s dB_s.$$

Exercise 7.3 (Hull-White interest rate model)

In the Hull-White model the dynamics of the overnight interest rate $r = \{r(t), t \in [0, T]\}$ are described by the following stochastic differential equation

$$r(t) = x + \int_0^t (a(s) - b(s)r(s))ds + \int_0^t \sigma(s) dB_s,$$

where $B = \{B_t, t \in [0, T]\}$ is a Brownian motion and a, b and σ are non-random positive functions of the time variable t.

Find the explicit solution to this equation by using the integration by parts formula from the lecture applied to the "integrating factor"

$$V(t) = \exp\left(\int_0^t b(s) ds\right)$$

and $Z(t) = r(t)$.

Exercise 7.4 (Vasicek model with jumps)

Suppose that the short rates $r(t)$ are modelled by the stochastic differential equation

$$r(t) = x + \int_0^t a(b - r(s))ds + \int_0^t \sigma dL_s,$$

where a, b and σ are non-negative constants and $L = \{L_t, t \in [0,T]\}$ is a Lévy process, that is $L_0 = 0$ a.s. and L has (as the Brownian motion) independent and stationary (but not necessarily normally distributed) increments. In addition, assume that L is a martingale with $E[|L_t|^2] < \infty$ for all $t \in [0,T]$.

Exercise 7.5

Let $X = \{X_t, t \geqslant 0\}$ be a regular time-homogeneous Markov chain as a model for stochastic interest rates and denote by $N_{jk}(t)$ the number of transitions from state j to state $k \neq j$ by time t.

Calculate the "speed" of changes of the expected number of interest rate transitions from j to k at time t, given $X_t = j$, that is

$$\frac{E[N_{jk}(t+h) - N_{jk}(t) | X_t = j]}{h}$$

for $h \searrow 0$ by using the following fact (which can be used for an alternative definition of Markov chains X_t): Consider the *jump chain* of X_t:

$$Y_n := X_{J_n},$$

where J_n is the n-th jump time of X_t. Then Y_n, $n \geqslant 0$ is a Markov chain with transition probabilities

$$p_{ij} = \begin{cases} \mu_{ij}/\mu_i, & j \neq i \text{ and } \mu_i \neq 0, \\ 0, & j \neq i \text{ and } \mu_i = 0 \end{cases} \qquad p_{ii} = \begin{cases} 0, & \mu_i \neq 0, \\ 1, & \mu_i = 0 \end{cases}$$

where μ_{ij} are the transition rates of X_t. Moreover, for all $n \geqslant 1$, $i_0, \ldots, i_{n-1}$, conditional on $Y_0 = i_0, \ldots, Y_{n-1} = i_{n-1}$ the holding times $S_j := J_j - J_{j-1}$, $j = 1, \ldots, n$ ($J_0 = 0$) are independent and exponentially distributed with parameters $\mu_{i_0}, \ldots, \mu_{i_{n-1}}$.