STK4500: Life Insurance and Finance

Exercise list 7: Solutions

Exercise 7.1

Let $f : [0, T] \to \mathbb{R}$ be a continuous function and $B = \{B_t, t \in [0, T]\}$ a Brownian motion. Show that

$$\int_0^T f(s) dB_s$$

is normally distributed with mean zero and variance given by

$$\int_0^T f(s)^2 ds.$$

Solution:

We can find step functions f_n such that

$$f_n(s) \xrightarrow{n\to\infty} f(s)$$

for all s with $|f_n(s)| \leqslant C$ for all n, s (C constant). On the other hand, we know from Itô's isometry that

$$\mathbb{V}ar\left[\int_0^T (f_n(s) - f(s)) dB_s\right] = \mathbb{E}\left[\int_0^T (f_n(s) - f(s))^2 ds\right] \xrightarrow{n\to\infty} 0,$$

which implies

$$\int_0^T f_n(s) dB_s \xrightarrow{n\to\infty} \int_0^T f(s) dB_s$$

in the sense of variance or, in particular, in probability.

Therefore, we may without loss of generality assume that f is a step function given by

$$f(s) = \sum_{i=1}^{n-1} a_i \mathbf{1}_{(t_i, t_{i+1}]}(s),$$

where $0 = t_0 < t_1 < \cdots < t_n = T$. By the definition of stochastic integrals

$$\int_0^T f(s) dB_s = \sum_{i=1}^{n-1} a_i (B_{T\wedge t_{i+1}} - D_{T\wedge t_i}) = \sum_{i=1}^{n-1} a_i (\underbrace{B_{t_{i+1}} - B_{t_i}}_{=:\xi_i}).$$

Now we have that $\xi_i \sim N(0, t_{i+1} - t_i)$ and ξ_i, $i = 1, \dots, n-1$ are independent because of the properties of the Brownian motion. Then

$$\int_0^T f(s) dB_s$$

is normally distributed with

$$\mathbb{E}\left[\int_0^T f(s) dB_s\right] = \sum_{i=1}^{n-1} a_i E[\xi_i] = 0$$

and

$$\mathbb{V}ar\left[\int_0^T f(s) dB_s\right] \overset{\text{indep.}}{=} \sum_{i=1}^{n-1} a_i^2 \mathbb{V}ar\left[\xi_i\right] = \sum_{i=1}^{n-1} a_i^2 (t_{i+1} - t_i) = \int_0^T f(s)^2 ds.$$

Exercise 7.2

Let $B = \{B_t, t \in [0, T]\}$ be a Brownian motion.

(i) Compute $[B, B]_t$ using the definition of quadratic variation.

(ii) Use (i) to evaluate

$$\int_0^T B_s dB_s.$$

Solution:

Let $0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_{m_n}^{(n)} = T$ be a sequence of partitions of $[0, T]$ with mesh $\max_{i=1,\dots,n} |t_i^{(n)} - t_{i-1}^{(n)}| \to 0$ as $n \to \infty$.

Define

$$f_n(s) = \sum_{i=0}^{m_n} B_{t_i^{(n)}} \mathbf{1}_{(t_i^{(n)}, t_{i+1}^{(n)}]}(s)$$

a stochastic step function. Then $f_n(s) \to B_s$ for all $s \in (0, T]$. Itô's isometry implies

$$\begin{aligned}
\mathbb{V}ar\left[\int_0^T (f_n(s) - B_s) dB_s\right] &= \mathbb{E}\left[\int_0^T (f_n(s) - B_s)^2 ds\right] \\
&= \mathbb{E}\left[\int_0^T (f_n(s)^2 - 2B_s f_n(s) + B_s^2) ds\right] \\
&= \int_0^T \mathbb{E}[f_n(s)^2] ds - 2\int_0^T \mathbb{E}[B_s f_n(s)] ds + \int_0^T \underbrace{\mathbb{E}[B_s^2]}_{=s} ds \\
&= \underbrace{\sum_{i=0}^{m_n - 1} \underbrace{\mathbb{E}[B^2_{t_i^{(n)}}]}_{=t_i^{(n)}} \mathbf{1}_{(t_i^{(n)}, t_{i-1}^{(n)}]}(s)}_{\to s} - 2 \underbrace{\mathbb{E}[B_s f_n(s)]}_{\sum_{i=0}^{m_n} (s \wedge t_i^{(n)}) \mathbf{1}_{(t_i^{(n)}, t_{i+1}^{(n)}]}(s) \to s} \int_0^T s ds
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^T (s - 2s + s) ds \\
&= 0.
\end{aligned}
$$

Thus

$$
\int_0^T f_n(s) dB_s = \sum_{i=0}^{m_n-1} B_{t_i^{(n)}} (B_{t_{i+1}^{(n)}} - B_{t_i^{(n)}}) \xrightarrow{n\to\infty} \int_0^T B_s dB_s
$$

in the sense of variance, or in particular, probability.

On the other hand, using the telescopic sum, we get

$$
\sum_{i=0}^{m_n-1} B_{t_i^{(n)}} (B_{t_{i+1}^{(n)}} - B_{t_i^{(n)}}) + \frac{1}{2} \sum_{i=0}^{m_n-1} (B_{t_{i+1}^{(n)}} - B_{t_i^{(n)}})^2 = \frac{1}{2} \sum_{i=0}^{m_n-1} (B^2_{t_{i+1}^{(n)}} - B^2_{t_i^{(n)}}) = \frac{1}{2} B_T^2.
$$

However,'

$$
\begin{aligned}
\mathbb{E}\left[\left(\sum_{i=0}^{m_n-1} (B_{t_{i+1}^{(n)}} - B_{t_i^{(n)}})^2 - T\right)^2\right] &= \mathbb{E}\left[\sum_{i,j=0}^{m_n-1} (B_{t_{i+1}^{(n)}} - B_{t_i^{(n)}})^2 (B_{t_{j+1}^{(n)}} - B_{t_j^{(n)}})^2\right] \\
&\quad - 2T \sum_{i=0}^{m_n-1} \underbrace{\mathbb{E}\left[(B_{t_{i+1}^{(n)}} - B_{t_i^{(n)}})^2\right]}_{\sum_{i=0}^{m_n-1} |t_{i+1}^{(n)} - t_i^{(n)}| = T} + T^2 \\
&\overset{\text{indep.}}{=} \sum_{i,j=0}^{m_n-1} \underbrace{\mathbb{E}\left[(B_{t_{i+1}^{(n)}} - B_{t_i^{(n)}})^2\right]}_{= t_{i+1}^{(n)} - t_i^{(n)}} \underbrace{\mathbb{E}\left[(B_{t_{j+1}^{(n)}} - B_{t_j^{(n)}})^2\right]}_{= t_{j+1}^{(n)} - t_j^{(n)}} \\
&\quad + \sum_{i=0}^{m_n-1} \underbrace{E\left[(\overbrace{B_{t_{i+1}^{(n)}} - B_{t_i^{(n)}}}^{N(0, t_{i+1}^{(n)} - t_i^{(n)})})^4\right]}_{3(t_{i+1}^{(n)} - t_i^{(n)})^2} - T^2 \\
&= \left(\underbrace{\sum_{i=0}^{m_n-1} (t_{i+1}^{(n)} - t_i^{(n)})}_{=T}\right)^2 - T^2 + 2 \underbrace{\sum_{i=0}^{m_n-1} (t_{i+1}^{(n)} - t_i^{(n)})^2}_{\leqslant 2 \max_{i=1,\dots,m_n-1} |t_{i+1}^{(n)} - t_i^{(n)}| T} \\
&\leqslant 2 \max_{i=1,\dots,m_n-1} |t_{i+1}^{(n)} - t_i^{(n)}| T \xrightarrow{n\to\infty} 0.
\end{aligned}
$$

As a result,

$$
\sum_{i=0}^{m_n-1} (B_{t_{i+1}^{(n)}} - B_{t_i^{(n)}})^2 \xrightarrow{n\ to\infty} T
$$

with probability one (at least for a subsequence).

Note: $\mathbb{E}[(X_n - X)^2] \xrightarrow{n\to\infty} 0 \Rightarrow X_n \xrightarrow{n\to\infty} X$ with probability one, for a subsequence of X_n, $n \geqslant 1$.

We know that

$$\sum_{i=0}^{m_n-1} (B_{t_{i+1}^{(n)}} - B_{t_i^{(n)}})^2 \xrightarrow{n\ to\infty} [B,B]_T$$

with probability one, at least for a subsequence. Hence, $[B,B]_T = T$.

For the second item we have

$$\int_0^T B_s dB_s + \frac{1}{2}[B,B]_T = \frac{1}{2}B_T^2$$

and hence

$$\int_0^T B_s dB_s = \frac{1}{2}(B_T^2 - T).$$

Exercise 7.3 (Hull-White interest rate model)

In the Hull-White model the dynamics of the overnight interest rate $r = \{r(t), t \in [0,T]\}$ are described by the following stochastic differential equation

$$r(t) = x + \int_0^t (a(s) - b(s)r(s))ds + \int_0^t \sigma(s)dB_s,$$

where $B = \{B_t, t \in [0,T]\}$ is a Brownian motion and a, b and σ are non-random positive functions of the time variable t.

Find the explicit solution to this equation by using the integration by parts formula from the lecture applied to the "integrating factor"

$$V(t) = \exp\left(\int_0^t b(s)ds\right)$$

and $Z(t) = r(t)$.

<u>Solution</u>:

Use Itô's formula applied to $X_t^{(1)} = V(t)$, $X_t^{(2)} = r(t)$ and $f(x_1,x_2) = x_1x_2$ to get

$$\begin{aligned} V(t)r(t) = f(X_t^{(1)}, X_t^{(2)}) &= \underbrace{X_0^{(1)}X_0^{(2)}}_{=x} \\ &+ \int_0^t \frac{\partial f}{\partial x_1}(X_{s^-}^{(1)}, X_{s^-}^{(2)})dX_s^{(1)} + \int_0^t \frac{\partial f}{\partial x_2}(X_{s^-}^{(1)}, X_{s^-}^{(2)})dX_s^{(2)} \\ &+ \frac{1}{2}\sum_{1\leqslant i,j\leqslant 2}\int_0^t \frac{\partial^2 f}{\partial x_1 \partial x_2}(X_{s^-}^{(1)}, X_{s^-}^{(2)})d[X^{(i)}, X^{(j)}]_s^c \\ &\sum_{0<s\leqslant t}\left[\underbrace{f(X_s^{(1)}, X_s^{(2)}) - f(X_{s^-}^{(1)}, X_{s^-}^{(2)})}_{=0}\right] - \sum_{1\leqslant i,j\leqslant 2}\frac{\partial f}{\partial x_i}(X_{s^-}^{(1)}, X_{s^-}^{(2)})\underbrace{\Delta X_s^{(i)}}_{=0}, \end{aligned}$$

where we used that $s \mapsto X_s$ is a.s. continuous.

Recall that $X_t^{(1)}$, $X_t^{(2)}$ continuous in t $\Rightarrow$

$$[X^{(i)}, X^{(j)}]_t^c = [X^{(i)}, X^{(j)}]_t - X_0^{(i)} X_0^{(j)} = [X^{(i)} - X_0^{(i)}, X^{(j)}]_t \tag{0.1}$$

' Note: $[A, D]_t = 0$ if A or D are of bounded variation with continuous paths starting in zero (See List 8). Since $\frac{d}{dt}V(t) = b(t)V(t)$ a.e. we know that

$$V(t) = 1 + \int_0^t b(s)V(s)ds$$

then V is of bounded variation (as an integral w.r.t. ds) and with continuous paths. Hence,

$$[X^{(i)}, X^{(j)}]_t^c = 0 \Rightarrow V(t)r(t) = x + \int_0^t X_s^{(2)} dX_s^{(1)} + \int_0^t X_s^{(1)} dX_s^{(2)}.$$

Furthermore,

$$\begin{aligned} V(t)r(t) &= x + \int_0^t r(s)b(s)V(s)ds + \int_0^t V(s)(a(s) - b(s)r(s))ds + \int_0^t V(s)\sigma(s)dB_s \\ &= x + \int_0^t V(s)a(s)ds + \int_0^t V(s)\sigma(s)dB_s. \end{aligned}$$

As a result

$$r(t) = e^{-\int_0^t b(s)ds} x + e^{-\int_0^t b(s)ds} \int_0^t V(s)a(s)r(s)ds + e^{-\int_0^t b(s)ds} \int_0^t V(s)\sigma(s)dB_s$$

which gives us the explicit unique solution. Alternatively, one can also use the integration by parts formula from Definition 7.9 to obtain the result.

Exercise 7.4 (Vasicek model with jumps)

Suppose that the short rates $r(t)$ are modelled by the stochastic differential equation

$$r(t) = x + \int_0^t a(b - r(s))ds + \int_0^t \sigma dL_s,$$

where a, b and σ are non-negative constants and $L = \{L_t, t \in [0, T]\}$ is a Lévy process, that is $L_0 = 0$ a.s. and L has (as the Brownian motion) independent and stationary (but not necessarily normally distributed) increments. In addition, assume that L is a martingale with $E[|L_t|^2] < \infty$ for all $t \in [0, T]$.

Solution:

Define $V(t) = e^{at}$ then $V(t) = 1 + \int_0^t aV(s)ds$. Thus V is of bounded variation (as an integral w.r.t. ds) and continuous. Hence, by similar computations as in the previous exercise we have

$$[V, r]_t = \underbrace{[V - \overbrace{V(0)}^{=1}, r]}_{=0 \text{ (because B.V.)}} + V(0)r(0) = V(0)r(0).$$

Using the integration by parts formula from Definition 7.9 applied to $V(t)$ and $r(t)$ we get

$$\underbrace{[V,r]_t}_{=V(0)r(0)} = V(t)r(t) - \int_0^t \underbrace{V(s^-)}_{V(s)}\,dr(s) - \int_0^t \underbrace{r(s^-)}_{r(0^-):=0}\,dV(s).$$

Hence,

$$V(t)r(t) = x + \int_0^t V(s)dr(s) + \int_0^t r(s^-)dV(s).$$

Substituting the differentials $dr(s)$ and $dV(s)$ we have

$$V(t)r(t) = x + \int_0^t V(s)a(b-r(s))ds + \int_0^t V(s)\sigma dL_s + \underbrace{\int_0^t r(s^-)aV(s)ds.}_{=\int_0^t r(s)aV(s)ds}$$

since $r(s)$ only has countably many jumps

Now $\int_0^t V(s)abds = b(e^{at}-1)$ and as a consequence

$$r(t) = xe^{-at} + b(1-e^{-at}) + \int_0^t e^{-a(t-s)}\sigma dL_s.$$

Exercise 7.5

Let $X = \{X_t, t \geqslant 0\}$ be a regular time-homogeneous Markov chain as a model for stochastic interest rates and denote by $N_{jk}(t)$ the number of transitions from state j to state $k \neq j$ by time t.

Calculate the "speed" of changes of the expected number of interest rate transitions from j to k at time t, given $X_t = j$, that is

$$\frac{E[N_{jk}(t+h) - N_{jk}(t)|X_t = j]}{h}$$

for $h \searrow 0$ by using the following fact (which can be used for an alternative definition of Markov chains X_t): Consider the *jump chain* of X_t:

$$Y_n := X_{J_n},$$

where J_n is the n-th jump time of X_t. Then Y_n, $n \geqslant 0$ is a Markov chain with transition probabilities

$$p_{ij} = \begin{cases} \mu_{ij}/\mu_i, & j \neq i \text{ and } \mu_i \neq 0, \\ 0, & j \neq i \text{ and } \mu_i = 0 \end{cases} \qquad p_{ii} = \begin{cases} 0, & \mu_i \neq 0, \\ 1, & \mu_i = 0 \end{cases}$$

where μ_{ij} are the transition rates of X_t. Moreover, for all $n \geqslant 1$, $i_0, \ldots, i_{n-1}$, conditional on $Y_0 = i_0, \ldots, Y_{n-1} = i_{n-1}$ the holding times $S_j := J_j - J_{j-1}$, $j = 1, \ldots, n$ $(J_0 = 0)$ are independent and exponentially distributed with parameters $\mu_{i_0}, \ldots, \mu_{i_{n-1}}$.

Solution:

Since X_s, $s \geqslant 0$ is a time-homogeneous process we can set $t = 0$ in

$$\mathbb{E}\left[N_{jk}(t+h) - N_{jk}(t)|X_t = j\right]/h.$$

It follows from the definition of $N_{jk}(s)$, $s \geqslant 0$ that

$$N_{jk}(h) = \sum_{n\geqslant 1} \mathbf{1}_{\{\underbrace{X_{J_{n-1}}}_{=Y_n}=j,\, \underbrace{X_{J_n}}_{=Y_{n-1}}=k, J_n<h\}}.$$

Hence.

$$\mathbb{E}\left[N_{jk}(h)|X_0=j\right] = \sum_{n\geqslant 1} \mathbb{E}\left[\mathbf{1}_{\{S_0+1S_1+\cdots,S_n<h\}}\mathbf{1}_{\{Y_{n-1}=j\}}\mathbf{1}_{\{Y_n=k\}}\mathbf{1}_{\{Y_0=j\}}\right]\frac{1}{\mathbb{P}(Y_0=j)}$$

$$= \sum_{n\geqslant 1}\sum_{m_1,\ldots,m_{n-2}\in S} \mathbb{E}\left[\mathbf{1}_{\{S_0+1S_1+\cdots,S_n<h\}}\mathbf{1}_{\{Y_n=k,Y_{n-1}=j,Y_{n-2}=m_{n-2},\ldots,Y_1=m_1,Y_0=j\}}\right]\frac{1}{\mathbb{P}(Y_0=j)}$$

$$= \sum_{n\geqslant 1}\sum_{m_1,\ldots,m_{n-2}\in S} \mathbb{E}\left[\overbrace{\mathbf{1}_{\{S_0+1S_1+\cdots,S_n<h\}}}^{\substack{\text{indep. and}\\ \text{exp. distributed}}}|Y_0=j,Y_1=m_1,\ldots,Y_{n-2}=m_{n-2},Y_n=j,Y_n=k\right]$$

$$\times\frac{1}{\mathbb{P}(Y_0=j)}\mathbb{P}(Y_0=j,Y_1=m_1,\ldots,Y_{n-1}=j,Y_n=k).$$

Generalized Erlang distribution for sums of independent exponentially distributed random variables (use induction):

$$\mathbb{E}\left[\mathbf{1}_{\{S_0+1S_1+\cdots,S_n<h\}}|Y_0=j,Y_1=m_1,\ldots,Y_{n-2}=m_{n-2},Y_n=j,Y_n=k\right]$$
$$=\mathbb{P}(S_0+S_1+\cdots+S_n<h|Y_0=j,Y_1=m_1,\ldots,Y_{n-2}=m_{n-2},Y_n=j,Y_n=k)=1-\alpha e^{h\Theta}\mathbb{I},$$

where

$$\Theta_n := \begin{pmatrix} -\mu(j) & \mu(j) & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mu(m_1)\mu(m_1) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\mu(m_{n-2})\mu(m_{n-2}) \\ 0 & 0 & 0 & 0 & 0 & 0 & -\mu(j) \end{pmatrix}$$

for $\alpha := (1,0,\ldots,0)$ and $\mathbb{I}=(1,\ldots,1)^t$. Then by the mean value theorem we have

$$1-\alpha e^{h\Theta_n}\mathbb{I} = \int_0^t -\alpha e^{th\Theta_n}\Theta_n\mathbb{I}dth.$$

$\Rightarrow$

$$\mathbb{E}\left[N_{jk}(h)|X_s=j\right]/h =$$
$$\sum_{n\geqslant 1}\sum_{m_1,\ldots,m_{n-2}\in S}\int_0^1(-\alpha e^{th\Theta_n}\Theta_n\mathbb{I})dt\; h\frac{1}{\mathbb{P}(Y_0=j)}\mathbb{P}(Y_0=j,Y_1=m_1,\ldots,Y_{n-1}=j,Y_n=k)$$
$$=\sum_{n\geqslant 1}\int_0^1 \underbrace{(-\alpha e^{th\Theta_n}\Theta_n\mathbb{I})}_{\sum_{\mu\geqslant 0}h^\mu\left(\frac{t^\mu}{\mu!}(-\alpha\Theta_n^{\mu+1}\mathbb{I})\right)}\; dt\;\underbrace{\frac{1}{\mathbb{P}(Y_0=j)}\mathbb{P}(Y_0=j,Y_1=m_1,\ldots,Y_{n-1}=j,Y_n=k)}_{=\mathbb{P}(Y_n=k|Y_0=j,Y_{n-1}=j)\mathbb{P}(Y_{n-1}=j|Y_0=j)=p_{jk}P(Y_{n-1}=j|Y_0=j)}$$

$$= \sum_{\mu \geqslant 0} \frac{h^\mu}{\mu!} \int_0^t t^\mu dt \sum_{n \geqslant 1} (-\alpha \Theta_n^{\mu+1} \mathbb{I}) p_{jk} \mathbb{P}(Y_{n-1} = j | Y_0 = j).$$

Letting $h \searrow 0$ we have

$$\xrightarrow{h \searrow 0} \sum_{n \geqslant 0} \underbrace{(-\alpha \Theta_n \mathbb{I})}_{\substack{\mu_j \text{ if } n=1 \\ 0 \text{ if } n \geqslant 1}} \underbrace{p_{jk}}_{= \frac{\mu_{jk}}{\mu_j}} \mathbb{P}(Y_{n-1} = j | Y_0 = j) = \mu_{jk}.$$