STK4500: Life Insurance and Finance

Exercise list 8: Solutions

Exercise 8.1

Consider the process

$$M(t) := \mathbb{E}[V|\mathcal{F}_t], \quad t \geqslant 0,$$

where $\mathcal{F} = \{\mathcal{F}_t\}_{t\geqslant 0}$ is a filtration (as a model for information flow over time) and V is a random variable (e.g. present value of the insurer's liabilities) with $E[|V|] < \infty$. Verify that M is a martingale with respect to $\mathcal{F}$.

Solution:

Obviously $\mathbb{E}[V|\mathcal{F}_t]$ finite for every t since by Jensen's inequality and the tower property we have $\mathbb{E}[|\mathbb{E}[V|\mathcal{F}_t]|] \leqslant \mathbb{E}[|V|] < \infty$. Also, $\mathbb{E}[V|\mathcal{F}_t]$ is $\mathcal{F}_t$-measurable for every t by definition. It remains to show the martingale property which follows by the tower property. Indeed, define $M_t \triangleq \mathbb{E}[V|\mathcal{F}_t]$, then for $s < t$ we have

$$\mathbb{E}[M_t|\mathcal{F}_s] = \mathbb{E}[\mathbb{E}[V|\mathcal{F}_t]|\mathcal{F}_s] = \mathbb{E}[V|\mathcal{F}_s] = M_s$$

since the σ-algebra $\mathcal{F}_s$ is smaller than $\mathcal{F}_t$ by the definition of filtration.

Exercise 8.2 (Generalized Black-Scholes model)

Let $Z = \{Z_t, t \in [0,T]\}$ be "market noise" modelled by a semimartingale with continuous paths. Assume that the price S_t of a stock at time $t \in [0,T]$ is described by the following stochastic differential equation

$$S_t = S_0 + \int_0^t S_u dZ_u, \quad t \in [0,T].$$

(i) Find an explicit formula for the stock price process S_t, $t \in [0,T]$ by using Itô's formula.

Solution:

Define the process $L_t = x\exp\left(Z_t - \frac{1}{2}[Z,Z]_t\right)$, $0 \leqslant t \leqslant T$, $Z_0 = 0$. Use Itô's formula applied to $X_t = Z_t - \frac{1}{2}[Z,Z]_t$ and $f(y) = \exp(y)$. Note that $\Delta X_t = 0$ since X is $\mathbb{P}$-a.s. continuous. Hence,

$$L_t = f(X_t) = \underbrace{f(X_0)}_{f(L_0)} + \int_0^t \underbrace{f'(X_s)}_{L_s} dX_s + \frac{1}{2}\int_0^t \underbrace{f''(X_s)}_{L_s} d\underbrace{[X,X]_s^c}_{[X,X]_s^c}.$$

Using the definition of $[X,Y]_t$ show that $[X,Y]$ is linear w.r.t. X and Y, hence

$$[X,X]_t = \left[Z - \frac{1}{2}[Z,Z], Z - \frac{1}{2}[Z,Z]\right] = [Z,Z]_t - \frac{1}{2}[[Z,Z],Z]_t - \frac{1}{2} - \frac{1}{2}[Z,[Z,Z]]_t + \frac{1}{4}[[Z,Z],[Z,Z]]_t.$$

Since $[Z,Z]_t$ is of bounded variation and $Z_0 = 0$, only the first term above remains. That is

$$[X,X]_t = [Z,Z]_t.$$

Hence,

$$L_t = L_0 + \int_0^t L_s dZ_s - \frac{1}{2}\int_0^t L_s d[Z,Z]_s + \frac{1}{2}\int_0^t L_s d\underbrace{[X,X]_s}_{[Z,Z]_s} = L_0 + \int_0^t L_s dZ_s.$$

By uniqueness we must have

$$S_t = x\exp\left(Z_t - \frac{1}{2}[Z,Z]_t\right).$$

(ii) Use the formula in (i) to obtain a representation for S_t in the case $Z_t = \int_0^t \mu ds + \int_0^t \sigma dB_s$ (classical Black-Scholes model) where B is a Brownian motion, $\mu \in \mathbb{R}$ the mean return and $\sigma > 0$ the volatility.

Solution:

Because of linearity, we have for

$$A_t \triangleq \int_0^t \mu ds, \quad D_t \triangleq \int_0^t \sigma dB_s,$$

that

$$[Z,Z]_t = [A,A]_t + [B,D]_t + [D,A]_t + [D,D]_t = \sigma^2 t,$$

where the three first terms above are 0 since A is of bounded variation and $A_0 = 0$. Thus,

$$S_t = x\exp\left(Z_t - \frac{1}{2}[Z,Z]_t\right) = x\exp\left((\mu t + \sigma B_t) - \frac{1}{2}\sigma^2 t\right) = x\exp\left(\left(\mu - \frac{1}{2}\sigma^2 t\right)t + \sigma B_t\right).$$

Exercise 8.3

Let $N = \{N_t, t \in [0,T]\}$ be a Poisson process with intensity $\lambda > 0$. Compute $\int_0^t N_{s-} dN_s$.

Solution:

We know from the definition of quadratic variation (Definition 7.9) that

$$[N,N]_t = N_t^2 - 2\int_0^t N_{s-} dN_s.$$

On the other hand, it follows from the properties of quadratic variation (See under Definition 7.15) that

$$[N,N]_t = \underbrace{N_0^2}_{=0} + \sum_{0<s\leqslant t} (\Delta N_s)^2.$$

The Poisson process N_t is increasing and performs jumps of size 1,

$$\Delta N_s = \mathbb{I}_{\{\Delta N_s \neq 0\}} = \begin{cases} 1, \text{ if } \Delta N_s \neq 0 \\ 0 \text{ else} \end{cases}$$

Using that $\mathbb{I}_A^2 = \mathbb{I}_A$ we conclude that

$$[N, N]_t = \sum_{0 < s \leqslant t} \mathbb{I}_{\{\Delta N_s \neq 0\}} = \sum_{0 < s \leqslant t} \Delta N_s = N_t.$$

Thus

$$\int_0^t N_{s-} dN_s = \frac{1}{2}(N_t^2 - N_t).$$