STK4500: Life Insurance and Finance

Exercise list 9

Exercise 9.1

Assume that the dynamics of the price S_t of a stock at time $t \in [0, T]$ is described by the Black-Scholes model, that is

$$S_t = S_0 + \int_0^t S_u \mu du + \int_0^t S_u \sigma dB_u,$$

where $B = \{B_t, t \in [0, T]\}$ is a Brownian motion, $\mu \in \mathbb{R}$ and $\sigma > 0$.

(i) Determine the probability measure $\mathbb{Q}$ (i.e. equivalent martingale measure) under which the discounted stock price

$$\widetilde{S}_t := e^{-rt} S_t, \quad t \in [0, T]$$

for a risk free rate of interest r becomes a martingale with respect to the "market information flow" $\mathcal{G} = \{\mathcal{G}_t\}_{t \in [0,T]}$.

(ii) Challenge: Pricing theory is classically based on the concept of martingality (i.e. "fairness"), hence we seek equivalent measures (i.e. measures that keep extremely rare events) that make prices martingales when discounted w.r.t. a reference asset (usually a bank account, which is one of the safest investments) like in (i). However, an alternative way of pricing is to fix the physical measure $\mathbb{P}$ and rather find a different reference asset, say $G = \{G_t, t \in [0, T]\}$, that when used as discount factor, makes prices martingales under $\mathbb{P}$. Construct a portfolio with value G_t, $t \in [0, T]$ such that

$$\widehat{S}_t := \frac{S_t}{G_t}, \quad t \in [0, T]$$

is a martingale under $\mathbb{P}$. Use this fact to provide a pricing formula under the real world measure $\mathbb{P}$, instead of the one from the lectures which is under $\mathbb{Q}$. This approach to pricing is sometimes referred to as *benchmark* pricing approach as opposite to the *risk neutral* pricing approach.

Exercise 9.2 (Markov property of Black-Scholes stock prices)

Consider the stock price process $S = \{S_t, t \in [0, T]\}$. Use the properties of the Brownian motion to show that

$$\mathbb{E}[f(S_t)|\mathcal{F}_s] = E[f(S_t^{s,x})]|_{x=S_s}$$

for all bounded functions f, where $S_t^{s,x}$ satisfies the "shifted" stochastic differential equation

$$S_t^{s,x} = x + \int_s^t S_u \mu du + \int_s^t S_u \sigma dB_u, \quad 0 \leqslant s \leqslant t.$$

The property that $S_t^{0,x} = S_t^{s,S_s^{0,x}}$ (a.s.) for all $0 \leqslant s \leqslant t$ is known as *flow property*. It tells us that i if we travel at time 0 from x to time t to S_t, we will arrive at the same point by travelling at time 0 from x to an intermediate time s to an intermediate point S_s and then, at time s from S_s to time t will lead to S_t as well.

Exercise 9.3

An insurer offers a 10-year unit-linked term insurance (or guaranteed minimum death benefit) with a single premium to a life aged $x_0 = 55$. An initial expense deduction of 4% is charged and the rest of the premium is invested in an equity fund whose dynamics S_t of its values over time is described by the Black-Scholes model in Exercise 1, with $S_0 = 1$. Further, management charges are deducted on a daily basis from the insured's account at a rate of $\beta = 0.6\%$ per year (i.e. in the sense of a continuous deduction based on the discount factor $e^{-\beta t}$). If death occurs during the contract period a death benefit of 110% of the fund value is provided.

Suppose

(i) Makeham's law

$$\mu_{*\dagger}(t) = A + Bc^t,$$

with $A = 0.0001$, $B = 0.00035$ and $c = 1.075$. Or if you want, you can use Norwegian mortality data from https://www.ssb.no/dode (Table 2) and using the data to estimate A, B and c.

(ii) Risk free rate of interest $r = 5\%$ per year, continuously compounded.

(iii) Volatility $\sigma = 25\%$ per year of S_t.

Calculate the guaranteed minimum death benefit value at issue, that is compute the prospective reserve $V_{i,\mathcal{F}}^{+}(t, A)$ of the benefits at the initial time of the contract.