STK4500: Life Insurance and Finance

Exercise list 9: Solutions

Exercise 9.1

Assume that the dynamics of the price S_t of a stock at time $t \in [0, T]$ is described by the Black-Scholes model, that is

$$S_t = S_0 + \int_0^t S_u \mu du + \int_0^t S_u \sigma dB_u,$$

where $B = \{B_t, t \in [0, T]\}$ is a Brownian motion, $\mu \in \mathbb{R}$ and $\sigma > 0$.

(i) Determine the probability measure $\mathbb{Q}$ (i.e. equivalent martingale measure) under which the discounted stock price

$$\widetilde{S}_t := e^{-rt} S_t, \quad t \in [0, T]$$

for a risk free rate of interest r becomes a martingale with respect to the "market information flow" $\mathcal{G} = \{\mathcal{G}_t\}_{t \in [0,T]}$.

Solution: An important special case of Itô's formula is when

$$Y_t = Y_0 + \int_0^t K_s ds + \int_0^t H_s dB_s$$

and $f = f(t, x)$, $t \geqslant 0$, $x \in \mathbb{R}$ is a function with continuous partial derivatives up to first order in time, and second in space. Then

$$f(t, Y_t) = f(0, Y_0) + \int_0^t \frac{d}{ds} f(s, Y_s) ds + \int_0^t \frac{d}{dx} f(s, Y_s) dY_s + \frac{1}{2} \int_0^t \frac{d^2}{dx^2} f(s, Y_s) H_s^2 ds.$$

Taking $f(t, x) = e^{-rt} x$ and $K_s = \mu S_s$ and $H_s = \sigma S_s$ we have

$$\widetilde{S}_t = f(t, S_t) = x + \int_0^t (-r) \widetilde{S}_u du + \int_0^t \mu \widetilde{S}_u du + \int_0^t \sigma \widetilde{S}_u dB_u = x + \int_0^t (\mu - r) \widetilde{S}_u du + \int_0^t \sigma \widetilde{S}_u dB_u.$$

Choose in Girsanov's theorem $X_t = \frac{r-\mu}{\sigma}$. Define

$$\widetilde{B}_t \triangleq B_t - \int_0^t X_s ds = B_t - \frac{r - \mu}{\sigma} t.$$

Then $\widetilde{B}$ is a Brownian motion under $\widetilde{\mathbb{P}}$ given by

$$\widetilde{\mathbb{P}}(A) \triangleq \mathbb{E}\left[\mathbb{I}_A Z_T\right], \quad Z_T \triangleq \exp(\underbrace{\int_0^t X_s dB_s}_{\frac{r-\mu}{\sigma}B_t} - \frac{1}{2}\underbrace{\int_0^t X_s^2 ds}_{(\frac{r-\mu}{\sigma})^2 t}).$$

Idea: rewrite $\widetilde{S}_t$ in terms of the new Brownian motion $\widetilde{B}$ by substituting $B_t = \widetilde{B}_t + \frac{r-\mu}{\sigma}t$ (and using that $dB_t = d\widetilde{B}_t + \frac{r-\mu}{\sigma}dt$. Hence,

$$\widetilde{S}_t = x + \int_0^t \sigma \widetilde{S}_u d\widetilde{B}_u.$$

Since $\widetilde{B}$ is a Brownian motion under $\widetilde{\mathbb{P}}$ and $\widetilde{S}$ is adapted to the filtration generated by $\widetilde{B}$ and $\mathbb{E}_{\widetilde{\mathbb{P}}}\left[\int_0^T (\sigma\widetilde{S}_u)^2 du\right] < \infty$ (check) then $\widetilde{S}$ is a $\widetilde{\mathbb{P}}$-martingale. That is

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\left[\widetilde{S}_t \middle| \mathcal{G}_t\right] = \widetilde{S}_s, \quad t \geqslant s.$$

(ii) Challenge: Pricing theory is classically based on the concept of martingality (i.e. "fairness"), hence we seek equivalent measures (i.e. measures that keep extremely rare events) that make prices martingales when discounted w.r.t. a reference asset (usually a bank account, which is one of the safest investments) like in (i). However, an alternative way of pricing is to fix the physical measure $\mathcal{P}$ and rather find a different reference asset, say $G = \{G_t, t \in [0,T]\}$, that when used as discount factor, makes prices martingales under $\mathcal{P}$. Construct a portfolio with value G_t, $t \in [0,T]$ such that

$$\widehat{S}_t := \frac{S_t}{G_t}, \quad t \in [0,T]$$

is a martingale under $\mathcal{P}$. Use this fact to provide a pricing formula under the real world measure $\mathcal{P}$, instead of the one from the lectures which is under $\mathbb{Q}$. This approach to pricing is sometimes referred to as *benchmark* pricing approach as opposite to the *risk neutral* pricing approach.

Solution: Let $\theta_t = (\theta_t^0, \theta_t^1)$ be the amount invested in the riskless asset e^{rt} and S_t, respectively. The value of such portfolio is then given by

$$V_t^\theta = \theta_t^0 e^{rt} + \theta_t^1 S_t.$$

We assume that θ is self-financing, and hence the variations on V happen due to variations on the riskless and risky asset only. Hence,

$$dV_t^\theta = \theta_t^0 r e^{rt} dt + \theta_t^1 dS_t = \theta_t^0 r e^{rt} dt + \theta_t^1 S_t[\mu dt + \sigma dB_t] = [\theta_t^0 r e^{rt} + \theta_t^1 S_t]dt + \theta_t^1 \sigma S_t dB_t.$$

Observe that we can retrieve an SDE for V_t if we work with proportions rather than quantities. Define $\pi_t^0 \triangleq \frac{\theta_t^0 e^{rt}}{V_t}$ the proportion of wealth invested in the riskless asset and,

$\pi_t^1 \triangleq \frac{\theta_t^1 S_t}{V_t}$ the proportion of wealth invested in S_t. Obviously, $\pi_t^0 + \pi_t^1 = 1$ a.s. and we can recast the above expression as

$$dV_t^\theta = V_t^\theta \left[(r\pi_t^0 + \mu\pi_t^1)dt + \sigma\pi_t^1 dB_t \right]$$

Now we have an SDE for V_t^θ for any arbitraty strategy θ. Let us then apply Itô's formula to $\frac{S_t}{V_t}$ in order to see its dynamics under $\mathbb{P}$ (recall that we seek a, or *the*, strategy, say θ^* such that V^{θ^*} when used as discount factor makes prices martingales under $\mathbb{P}$). We will suppress the dependence on θ since it is now clear that we look at arbitrary strategies. Hence,

$$d\frac{S_t}{V_t} = d\left[\frac{1}{V_t}\right] S_t + \frac{1}{V_t} dS_t + d\left[\frac{1}{V_\cdot}, S_\cdot\right]_t. \tag{0.1}$$

For the factor $d\left[\frac{1}{V_t}\right]$ we need to apply Itô's formula again and use the SDE for V_t. Thus

$$d\left[\frac{1}{V_t}\right] = \frac{-1}{V_t^2} dV_t + \frac{1}{2} 2 \frac{1}{V_t^3} d\left[V_\cdot, V_\cdot\right]_t = -\frac{1}{V_t}\left[(r\pi_t^0 + \mu\pi_t^1 - \sigma^2(\pi_t^1)^2)dt + \sigma\pi_t^1 dB_t \right].$$

Now back to (0.1) we have

$$d\frac{S_t}{V_t} = -\frac{S_t}{V_t}\left[(r\pi_t^0 + \mu\pi_t^1 - \sigma^2(\pi_t^1)^2)dt + \sigma\pi_t^1 dB_t \right] + \frac{S_t}{V_t}[\mu dt + \sigma dB_t] - \frac{S_t}{V_t}\sigma^2\pi_t^1 dt.$$

Grouping terms of finite and infinite variation we have

$$d\frac{S_t}{V_t} = -\frac{S_t}{V_t}\left[(r\pi_t^0 + \mu\pi_t^1 - \sigma^2(\pi_t^1)^2 - \mu + \sigma^2\pi_t^1)dt + (\sigma\pi_t^1 - \sigma)dB_t \right]. \tag{0.2}$$

Now, denote $a \triangleq \pi_t^0$ and $b \triangleq \pi_t^1$ and imposing that the finite variation part is zero we obtain the following system of equations

$$\begin{aligned} ra + \mu b - \sigma^2 b^2 - \mu + \sigma^2 b &= 0 \\ a + b &= 1 \end{aligned}$$

which has the unique solution in $[0,1]^2$ given by

$$b = \frac{\mu - r}{\sigma^2}, \quad a = 1 - b.$$

Denote the value of the portfolio with strategy $\theta_t^* \triangleq (a, b)$ by $G_t \triangleq V_t^{\theta^*}$. Then we have proven that $\widehat{S}_t = \frac{S_t}{G_t}$, $t \geqslant 0$ is a $\mathbb{P}$-martingale.

You can also check that discounting the value or an arbitrary portfolio with a self-financing strategy θ with respect to G gives you a martingale under $\mathbb{P}$, thus providing a pricing formula under the real world measure $\mathbb{P}$ but...

OBS! Be very careful here, because this is only true if the processes involved in the stochastic intengrals, for example in the SDE (0.2), are in $L^2([0,T]\times\Omega)$. That is

$$\mathbb{E}\left[\int_0^T\left(\frac{S_t}{V_t}(\sigma\pi_t^1-\sigma)\right)^2dt\right]<\infty$$

This is true in this exercise (Black-Scholes setting) if the trategies are square-integrable. You can check that. But in general, if S is given by a general and more complicated SDE and θ is general, then you may find a portfolio G such that $\frac{S}{G}$ has 0 finite variation part, but this does not mean that the process is a martingale (we need square integrability as above). In general, we can only say that $\frac{S}{G}$ is a $\mathbb{P}$-local-martingale. Nevertheless, this is not enough for pricing. Recall that $\frac{S}{G}$ is lower-bounded. All local-martingales that are lower bounded are supermartingales, which means

$$\mathbb{E}\left[\frac{S_T}{G_T}\Big|\mathcal{G}_t\right]\leqslant\frac{S_t}{G_t}.$$

Similarly, the same happens with the discounted values of an arbitrary portoflio, that is $\frac{V_t}{G_t}$ may fail to be a martingale (only being a local-martingale) and hence, since it is lower-bounded we have that $\frac{V_t}{G_t}$ is a supermartingale, i.e.

$$\mathbb{E}\left[\frac{V_T}{G_T}\Big|\mathcal{G}_t\right]\leqslant\frac{V_t}{G_t}.$$

If V_T is the claim to be replicated, then you see that the price is bigger than the actual market value, leading to a "wrong" price.

The martingale property for pricing correctly is a very subtle thing. If discounted portfolios are not martingales (while we are assuming they are) this means that we are pricing "wrongly". There is a whole theory dealing with pricing under strict local-martingales. Such theory explains the existence of so-called bubbles. When an asset price bubble exists, the market price of the asset is higher than its fundamental value. From a mathematical point of view, this is the case when the stock price process is modeled by a positive strict local martingale under the equivalent local martingale measure.

Exercise 9.2 (Markov property of Black-Scholes stock prices)

Consider the stock price process $S=\{S_t,t\in[0,T]\}$. Use the properties of the Brownian motion to show that

$$\mathbb{E}[f(S_t)|\mathcal{F}_s]=E[f(S_t^{s,x})]|_{x=S_s}$$

for all bounded functions f, where $S_t^{s,x}$ satisfies the "shifted" stochastic differential equation

$$S_t^{s,x}=x+\int_s^t S_u\mu du+\int_s^t S_u\sigma dB_u,\quad 0\leqslant s\leqslant t.$$

The property that $S_t^{0,x}=S_t^{s,S_s^{0,x}}$ (a.s.) for all $0\leqslant s\leqslant t$ is known as *flow property*. It tells us that i if we travel at time 0 from x to time t to S_t, we will arrive at the same point by travelling at time 0 from x to an intermediate time s to an intermediate point S_s and then, at time s from S_s to time t will lead to S_t as well.

Exercise 9.3

An insurer offers a 10-year unit-linked term insurance (or guaranteed minimum death benefit) with a single premium to a life aged $x_0 = 55$. An initial expense deduction of 4% is charged and the rest of the premium is invested in an equity fund whose dynamics S_t of its values over time is described by the Black-Scholes model in Exercise 1, with $S_0 = 1$. Further, management charges are deducted on a daily basis from the insured's account at a rate of $\beta = 0.6\%$ per year (i.e. in the sense of a continuous deduction based on the discount factor $e^{-\beta t}$). If death occurs during the contract period a death benefit of 110% of the fund value is provided.

Suppose

(i) Makeham's law

$$\mu_{*\dagger}(t) = A + Bc^t,$$

with $A = 0.0001$, $B = 0.00035$ and $c = 1.075$. Or if you want, you can use Norwegian mortality data from https://www.ssb.no/dode (Table 2) and using the data to estimate A, B and c.

(ii) Risk free rate of interest $r = 5\%$ per year, continuously compounded.

(iii) Volatility $\sigma = 25\%$ per year of S_t.

Calculate the guaranteed minimum death benefit value at issue, that is compute the prospective reserve $V_{i,\mathcal{F}}^{+}(t, A)$ of the benefits at the initial time of the contract.

Solution: The policy function that defines entirely this contract is given by

$$a_{*\dagger}(t) = \begin{cases} C(t), & \text{if } 0 \leqslant t < 10, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$C(t) = (1 - 0.04) \cdot 1.10 \cdot P \cdot S_t e^{-0.006t}.$$

Here, $1 - 0.04$ is the portion of the premium P invested in the equity fund, S_t is the value of the fund at time t and $e^{-0.06t}$ is the discount factor for the management charges. We know from the definition of $V_{\mathcal{F}}^{+}(t, A)$ that

$$V_{\mathcal{F}}^{+}(t, A) = \int_t^{10} \pi_t^{*\dagger}(s) p_{**}(x + t, x + s) \mu_{*\dagger}(x + s) ds, \text{ if } X_t = *,$$

where $\pi_t^{*\dagger}(s)$ is the fair value at time t of $C(s)$.

Using the pricing formula (9.15) from the lecture notes, we have

$$\pi_t^{*\dagger}(s) = \text{ClaimValue}_t = \mathbb{E}_{\widetilde{\mathbb{P}}}\left[e^{-r(s-t)} C(s) \middle| \mathcal{G}_t \right],$$

where $\widetilde{\mathbb{P}}$ is the probability measure of Exercise 1 above. That is

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\left[\widetilde{S}_s \middle| \mathcal{G}_t \right] = \widetilde{S}_t, \quad s \geqslant t.$$

Altogether,

$$\begin{aligned}\pi_t^{*\dagger}(T) &= e^{rt}(1-0.04)\cdot 1.10\cdot P\cdot e^{-0.006s}\mathbb{E}_{\widetilde{\mathbb{P}}}\left[\widetilde{S}_s\middle|\mathcal{G}_t\right]\\ &= e^{rt}(1-0.04)\cdot 1.10\cdot P\cdot e^{-0.006s}\widetilde{S}_t.\end{aligned}$$

If $t=0$, then

$$\pi_0^{*\dagger}(s) = (1-0.04)1.10Pe^{-0.006s}\underbrace{\widetilde{S}_0}_{=1}.$$

As a result,

$$V_{\mathcal{F}}^{+}(0,A) = (1-0.04)\cdot 1.10\cdot P\underbrace{\int_0^{10} e^{-0.006s}p_{**}(55,55+s)\mu_{*\dagger}(55+s)ds}_{\approx 0.2329}$$

Here, recall that $p_{**}(t,s) = \exp\left(-\int_t^r \mu_{*\dagger}(u)du\right)$.

In conclusion, the prospective reserve at time $t=0$ of the death benefit is

$$V_{\mathcal{F}}^{+}(0,A) = 0.2459P \quad (= 2\,459\$ \text{ if e.g. } P = 10\,000\$)$$