UiO : Department of Mathematics
University of Oslo

Life Insurance and Finance

Lecture 1: Introduction and Markov modelling

David R. Banos

STK4500

Introduction of the course

STK4500/9500: Life Insurance and Finance

- **Webpage:** https://www.uio.no/studier/emner/matnat/math/STK4500/v23/index.html

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Life Insurance

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$$\text{State of the insured} \Longleftrightarrow \text{triggers payments}$$

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State of the insured $\Longleftrightarrow$ triggers payments

We need to model:

- State of the insured
- Value of the money
- Market (when investing in it)

Solvency II

...is a Directive in European Union law that codifies and harmonises the EU insurance regulation. Primarily this concerns the amount of capital that EU insurance companies must hold to reduce the risk of insolvency. https://eur-lex.europa.eu/legal-content/EN/TXT/PDF/?uri=CELEX:02009L0138-20210630&from=EN

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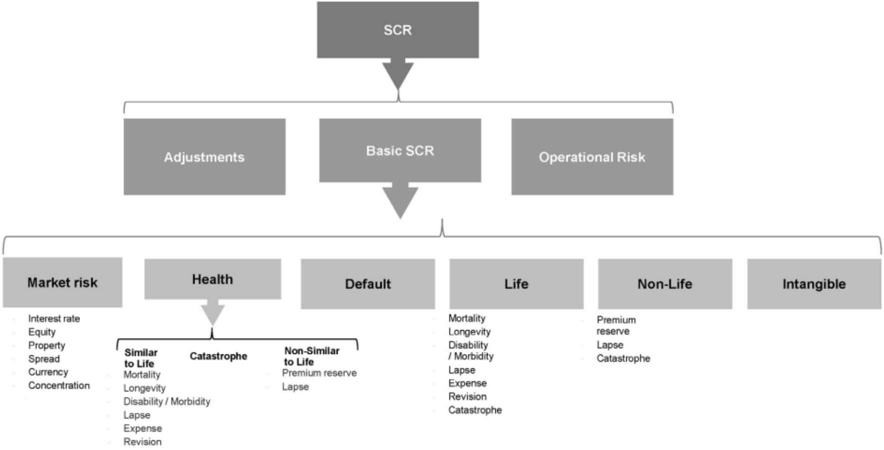
- See e.g. some articles from Section 4, page 87 onwards.
- See ANNEX IV, page 239, for solvency capital requirements.
- In this course we will mostly be dealing with SCR_{life}.

Comment: SCR_i in Solvency II are given as 99.5% *Value-at-Risk* quantiles of the loss distribution over one-year period. We will focus on the modelling and simulation of such loss distribution and its (conditional) expected value, also known as pure premium.


SCR
Adjustments
Basic SCR
Operational Risk
Market risk
Health
Default
Life
Non-Life
Intangible
Interest rate
Equity
Property
Spread
Currency
Concentration
Similar to Life
Catastrophe
Non-Similar to Life
Mortality
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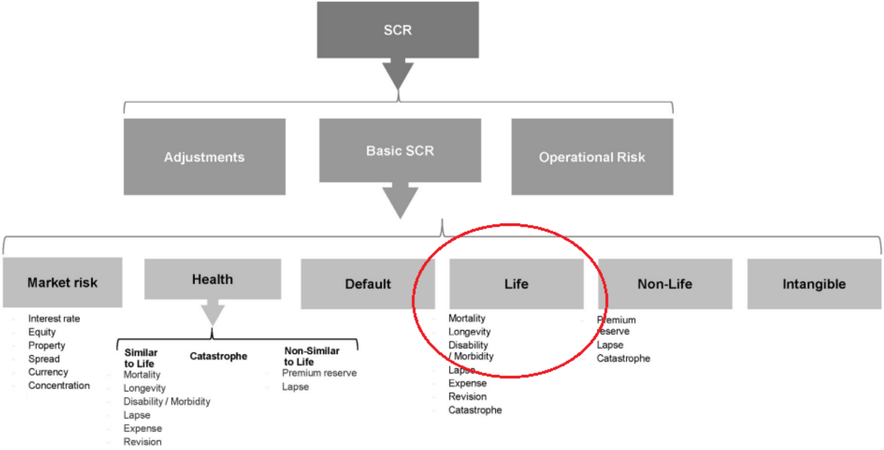
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Markov modelling

We use Markov processes to model the state of the insured

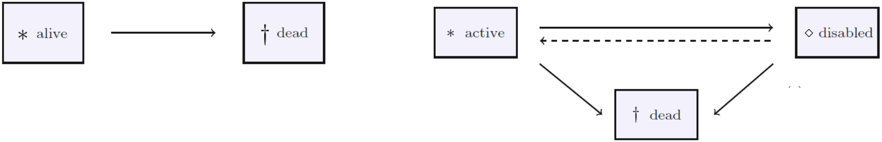

Figure: Survival model and disability model

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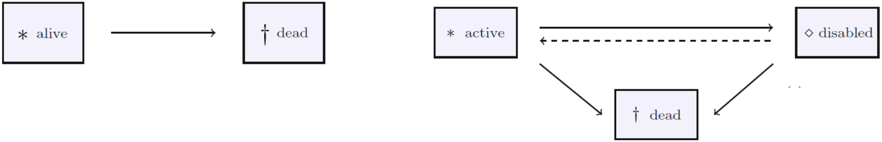

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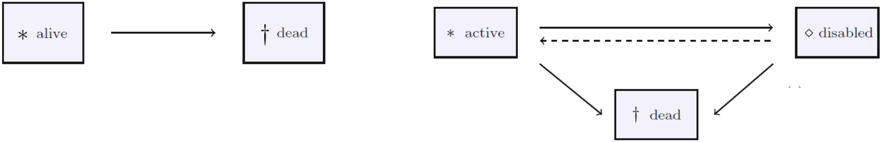

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- $\mathcal{S}$ denotes the state space. We assume $\mathcal{S}$ is countable, often finite. Example: $\mathcal{S} = \{*, \dagger\}$, $\mathcal{S} = \{*, \diamond, \dagger\}$, $\mathcal{S} = \{0, 1, 2, \ldots, n\}$.

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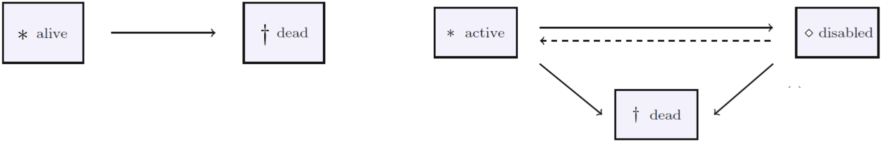

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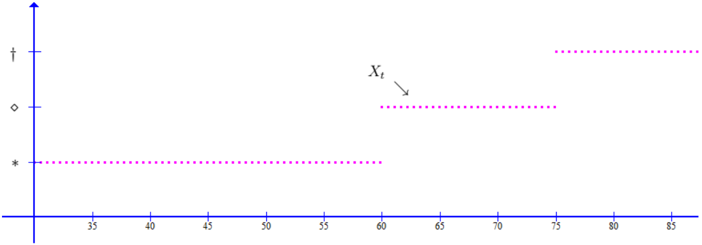
- $p_{ij}(t, s) \triangleq \mathbb{P}[X(s) = j | X(t) = i]$, $s, t \geq 0$, $t \leq s$, $i, j \in \mathcal{S}$ denotes the transition probabilities.



This is a way of illustrating the path $t \mapsto X_t(\omega)$ but the order in $\mathcal{S} = \{*, \diamond, \dagger\}$ is arbitrary. For the given outcome, say ω, in the figure, we have an individual who passed away at the age of 75, being inactive from age 60 to 75. In the insurance context, this outcome has a specific insurance loss determined by the policy of the individual.

Definition (Markov chain)

Let $X_t \in \mathcal{S}$, $t \in J \subseteq \mathbb{R}$ be a stochastic process on $(\Omega, \mathcal{A}, \mathbb{P})$. Then X_t, $t \in J$ is called a Markov chain, if

$$\mathbb{P}(X_{t_{n+1}} = i_{n+1} | X_{t_1} = i_1, \ldots, X_{t_n} = i_n) = \mathbb{P}(X_{t_{n+1}} = i_{n+1} | X_{t_n} = i_n)$$

for all $t_1 < t_2 < \cdots t_{n+1} \in J$, $i_1, \ldots, i_{n+1} \in \mathcal{S}$ with $\mathbb{P}(X_{t_1} = i_1, \ldots, X_{t_n} = i_n) \neq 0$.

Remark

The process $X_{t_{n+1}}$ at time t_{n+1} just remembers its last position $X_{t_n} = i_n$. Popularly, one says that $\{X_t\}_{t \in J}$ is a process "without memory".

Definition (Transition probability matrix)

Let $P(t,s)$ be the matrix containing the entries given by $p_{ij}(t,s)$, $i,j \in \mathcal{S}$ for $t,s \geq 0$, $t \leq s$. Then $P(t,s)$ is called the *transition probability matrix* of X.

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Conversely:

Definition (Transition probability matrix)

Let $P(t,s)$ be a matrix containing entries $p_{ij}(t,s)$, $i,j \in \mathcal{S}$ for $t,s \geq 0$, $t \leq s$. Then $P(t,s)$ is a *transition probability matrix* if, and only if,

1. $p_{ij}(t,s) \geq 0$.
2. $\sum_{j\in\mathcal{S}} p_{ij}(t,s) = 1$ for all $i \in \mathcal{S}$.
3. $p_{ij}(t,t) = \mathbf{1}_{\{i=j\}}$ provided that $\mathbb{P}[X_t = i] \neq 0$.

Theorem (Chapman-Kolmogorov equation)

Let $\{X_t\}_{t\in J}$, be a Markov chain and $P(s,t) = \{p_{ij}(s,t)\}_{i,j\in\mathcal{S}}$ its matrix of transition probabilities. Then

$$p_{ij}(s,t) = \sum_{k\in\mathcal{S}} p_{ik}(s,u)p_{k,j}(u,t),$$

for all $s \leq u \leq t$ and $i,j \in \mathcal{S}$ with $\mathbb{P}(X_s = i), \mathbb{P}(X_t = j) \neq 0$. Equivalently, in matrix notation

$$P(s,t) = P(s,u)P(u,t), \quad s \leq u \leq t.$$

Exercise

Prove the theorem above. **Hint:** *pick a middle time u between s and t and use the law of total probability and then the definition of conditional expectation.*

Theorem (Characterization of Markov chains)

A stochastic process $X = \{X_t\}_{t\in J}$, $J \subseteq \mathbb{R}$ is a Markov chain if, and only if

$$\mathbb{P}(X_{t_1} = i_1, \ldots, X_{t_n} = i_n) = \mathbb{P}(X_{t_1} = i_1) p_{i_1,i_2}(t_1, t_2) p_{i_2,i_3}(t_2, t_3) \cdots p_{i_{n-1},i_n}(t_{n-1}, t_n)$$

for all $t_1 < t_2 < \cdots < t_n \in J$, $i_1, \ldots, i_n \in \mathcal{S}$, $n \geq 1$.

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for all $t_1 < t_2 < \cdots < t_n \in J$, $i_1, \ldots, i_n \in \mathcal{S}$, $n \geq 1$.

Theorem (Markov property)

Let $t_1 < t_2 < \cdots < t_n < t_{n+1} < \cdots < t_{n+m}$, $i \in \mathcal{S}$, $A \subset \mathcal{S}^{n-1}$, $B \subset \mathcal{S}^m$. Assume that

$$\mathbb{P}\left((X_{t_1}, X_{t_2}, \ldots, X_{t_{n-1}}) \in A, X_{t_n} = i)\right) \neq 0.$$

Then the Markov property holds, that is

$$\begin{aligned} &\mathbb{P}\left((X_{t_{n+1}}, X_{t_{n+2}}, \ldots, X_{t_{n+m}}) \in B \mid (X_{t_1}, X_{t_2}, \ldots, X_{t_{n-1}}) \in A, X_{t_n} = i\right) && (1) \\ &= \mathbb{P}\left((X_{t_{n+1}}, X_{t_{n+2}}, \ldots, X_{t_{n+m}}) \in B | X_{t_n} = i\right) && (2) \end{aligned}$$

Definition (Homogeneous Markov chain)

A Markov chain $X = \{X_t\}_{t\in J}$ is called time homogeneous if

$$\mathbb{P}(X_{s+h} = j | X_s = i) = \mathbb{P}(X_{t+h} = j | X_t = i)$$

for all $s, t, h \geq 0$ and $i, j \in \mathcal{S}$, provided that $\mathbb{P}(X_s = i), \mathbb{P}(X_t = i) \neq 0$.

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Remark

(1) In matrix form: $P(s, s+h) = P(t, t+h) = P(0, h)$ and hence the transition probability matrix depends only on one parameter.

(2) In life insurance modelling, a Markov process is hardly time-homogeneous. Think of why.

Example

A 30 year old person. $\mathcal{S} = \{1, 2, 3\}$, 1: healthy, 2: ill, 3: deceased. $X_n \in \mathcal{S}$, $n \in \mathbb{N}$ and

$$P(n, n+1) \equiv P = \begin{pmatrix} 0.85 & 0.1 & 0.05 \\ 0.6 & 0.3 & 0.1 \\ 0 & 0 & 1 \end{pmatrix}$$

Algorithm:

1. Obtain the $\mathcal{S} \times \mathcal{S}$ probability transition matrix P.
2. Set $t = 0$
3. Pick an initial state $X_t = i$. Here e.g. $X_0 = 1$ or $X_0 = *$.
4. For $t = 1, \ldots, T$:
   1. Obtain the row of P corresponding to the current state X_t.
   2. Generate X_{t+1} from a multinomial distribution with probability vector equal to the row we obtained above.

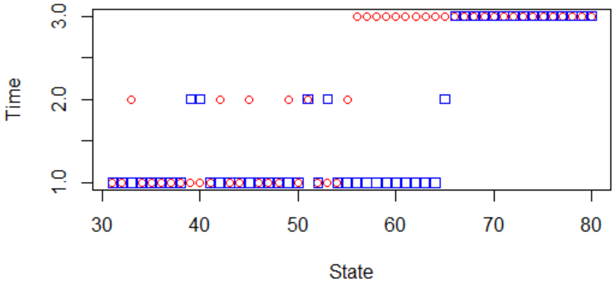

Figure: Two realizations of X_n, $n = 31, \ldots, 80$ with transition probability matrix P for $X_{30} = 1$ alive. The red outcome is a person who passed away at age 56 and the second outcome at age 66.

Transition rates

We now consider a continuous time Markov process X_t, $t \in \mathbb{R}$, $t \geq 0$ with finite state space $\mathcal{S}$.

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Definition (Transition rates)

Let $X = \{X_t, t \geq 0\}$ be a Markov process with finite state space $\mathcal{S}$. The transition rates μ_i, μ_{ij}, $i, j \in \mathcal{S}$, $j \neq i$ are the functions defined by

$$\mu_i(t) \triangleq \lim_{\substack{h \to 0 \\ h > 0}} \frac{1 - p_{ii}(t, t+h)}{h}, \quad t \geq 0, \quad i \in \mathcal{S}$$

and

$$\mu_{ij}(t) \triangleq \lim_{\substack{h \to 0 \\ h > 0}} \frac{p_{ij}(t, t+h)}{h}, \quad t \geq 0, \quad i, j \in \mathcal{S}, \quad j \neq i,$$

whenever they exist and are finite.

Definition (Regular Markov process)

Let $X = \{X_t, t \geq 0\}$ be a Markov process with finite state space $\mathcal{S}$. We say that X is *regular* if the transition rates μ_i, μ_{ij}, $i, j \in \mathcal{S}$, $j \neq i$ exist and are continuous as functions of t.

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We denote by $\Lambda(t)$ the transition rate matrix

$$\Lambda(t) = \begin{pmatrix} \mu_{11}(t) & \mu_{12}(t) & \cdots & \mu_{1n}(t) \\ \mu_{21}(t) & \mu_{22}(t) & \cdots & \mu_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n}(t) & \mu_{n2}(t) & \cdots & \mu_{nn}(t) \end{pmatrix},$$

where $\mu_{ii}(t) \triangleq -\mu_i(t)$ by convention.

Remark

(i) Observe that $\mu_{ij}(t) = \lim_{\substack{h\to 0\\ h>0}} \frac{p_{ij}(t,t+h)-p_{ij}(t,t)}{h} = \frac{d}{ds} p_{ij}(s,t)\Big|_{s=t+}$.

(iii) Interpretation: $\mu_{ij}(t)h \approx p_{ij}(t, t+h)$, $h>0$ small, which means $\mu_{ij}(t)h$ probability for switching from state i to state j on the infinitesimal interval $[t, t+h]$ and $\mu_{ij}(t)$ is the "speed".

(iv) Let $\Lambda(t) = \{\mu_{ij}(t)\}_{i,j\in S}$ be the transition rate matrix, $\mathcal{S} = \{1, \ldots, n\}$ and assume that X is homogeneous. Then $\Lambda(0)$ is the generator of the semigroup $P(t)$, $t \in J$, that is

$$\Lambda(0) = \lim_{h\searrow 0} = \frac{P(h) - Id_n}{h},$$

where Id_n denotes the $n \times n$ identity matrix.

(v) $P(t) = e^{\Lambda(0)t}$ (matrix exponential: $e^A \triangleq \sum_{n=0}^{\infty} \frac{1}{n!} A^n$ for a matrix A)

Theorem (Kolmogorov equations)

Let $X = \{X_t, t \geq 0\}$ be regular. Then

1. Backward Kolmogorov equation:

$$\frac{d}{ds}p_{ij}(s,t) = \underbrace{\mu_i(s)}_{\triangleq -\mu_{ii}(s)} p_{ij}(s,t) - \sum_{\substack{k \in \mathcal{S} \\ k \neq i}} \mu_{ik}(s) p_{kj}(s,t)$$

or in matrix notation

$$\frac{d}{ds}P(s,t) = -\Lambda(s)P(s,t) \text{ (matrix multiplication).}$$

Theorem (Kolmogorov equations)

2. Forward Kolmogorov equation:

$$\frac{d}{dt}p_{ij}(s,t) = -p_{ij}(s,t)\underbrace{\mu_j(t)}_{\triangleq -\mu_{jj}(t)} + \sum_{\substack{k\in\mathcal{S}\\ k\neq j}} p_{ik}(s,t)\mu_{kj}(t)$$

or in matrix notation

$$\frac{d}{dt}P(s,t) = P(s,t)\Lambda(t) \textit{ (matrix multiplication).}$$

Define

$$\overline{p}_{jj}(s,t) \triangleq \mathbb{P}\left(\bigcap_{\xi\in[s,t]}\{X_\xi = j\}|X_s = j\right) = \mathbb{P}\left(X_\xi = j \text{ for all } \xi \in [s,t]|X_s = j\right)$$

the probability that X stays in the state j during the time period $[s,t]$, given that $X_s = j$.

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the probability that X stays in the state j during the time period $[s,t]$, given that $X_s = j$.

Theorem (Calculation of $\overline{p}_{jj}(s,t)$)

If $X = \{X_t, t \geq 0\}$ is regular then

$$\overline{p}_{jj}(s,t) = \exp\left(-\sum_{k\neq j}\int_s^t \mu_{jk}(u)du\right)$$

provided that $\mathbb{P}(X_s = j) \neq 0$.

Example

A Markov process X_t with two states: $*$ alive and $\dagger$ deceased.

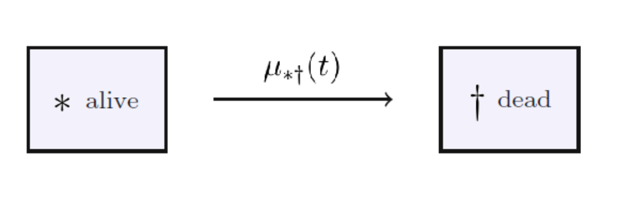

Figure: Survival model

Find the transition probabilities and simulate paths of such a process.

Some insurance Markov models

Survival model

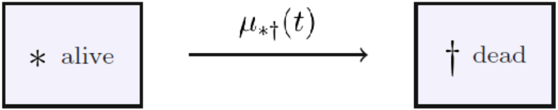

Figure: Survival model

The disability model

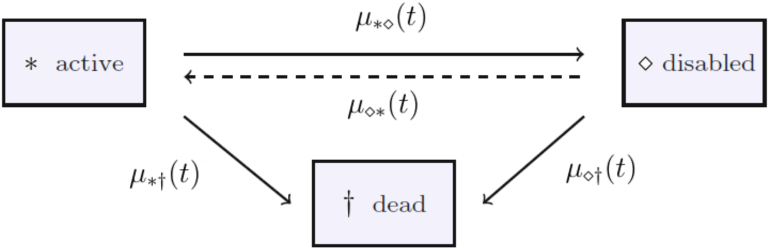

Figure: Disability model

The disability model with recovery

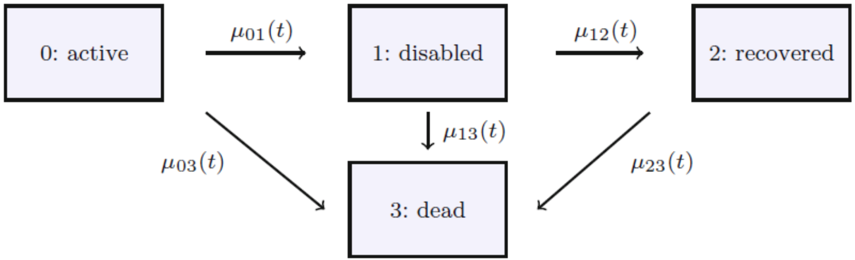

Figure: Disability model with possibility for recovery, where mortality is affected after recovery.

The spouse model

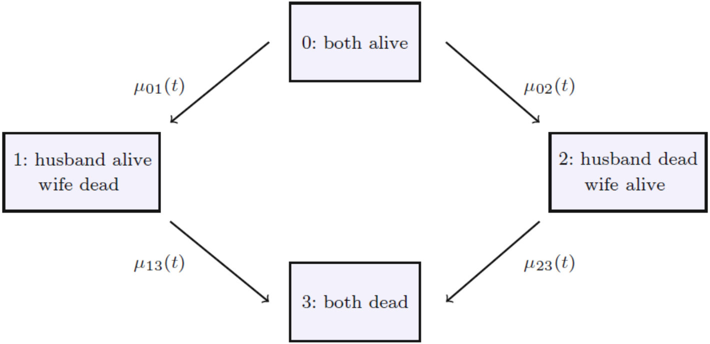

Figure: The spouse model with dependent lives.

The orphan pension model

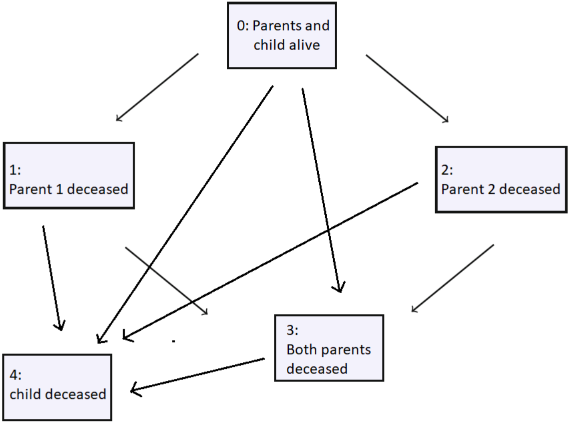

Figure: The orphan model.

Exercise (Competing risks model)

Consider an insurance policy that pays out a death benefit according to the type of death of the insured. We distinguish between deaths of the following types: natural, illness, accident, homicide and suicide. Draw a Markov diagram for this insurance and give expressions for the transition probabilities.

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Exercise (Disability with time dependent recovery rate)

In a disability insurance, it is unrealistic to assume that the probability of recovery is homogeneous with respect to disability length. It is the case that the probability of recovery decreases as the insured remains in state "disable". Consider a disability insurance with states $\mathcal{S} = \{*, \diamond, \dagger\}$ and split $\diamond$ into substates $\diamond_1, \ldots, \diamond_n$ for some fixed positive integer n, where state $\diamond_k$ is the state of a person being in its k-th disability year, $k = 1, \ldots, n$. Consider possibility of recovery in all states $\diamond_k$, $k = 1, \ldots, n-1$ except for $\diamond_n$ where recovery is no longer possible. Plot the Markov diagram of this insurance model and write down Kolmogorov equations for the transition probabilities.

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