UiO : Department of Mathematics
University of Oslo

Life Insurance and Finance

Lecture 10: Distributional properties of the stochastic prospective value

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STK4500

1. **Introduction**
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3. **Higher moments of V_t^+, $t \in \mathbb{N}$**
4. **Examples**
   - Distribution of endowment
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Introduction

Recall that the (stochastic) prospective value of a policy and the expected prospective value are different things. The former is a **random variable**, while the latter is the conditional **expectation**, given $X_t = i$, of the former.

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In this lecture we will confine ourselves to the **discrete time** setting

$$V_t^+ = \frac{1}{v(t)}\left[\sum_{n=t}^{\infty}\sum_{j} v(n) I_j^X(n) a_j^{\text{Pre}}(n) + \sum_{n=t}^{\infty}\sum_{j,k} v(n+1) a_{jk}^{\text{Post}}(n)\Delta N_{jk}^X(n)\right],$$

$$V_i^+(t) = \frac{1}{v(t)}\left[\sum_{n=t}^{\infty}\sum_{j} v(n) p_{ij}(t,n) a_j^{\text{Pre}}(n) + \sum_{n=t}^{\infty}\sum_{j,k} v(n+1) p_{ij}(t,n) p_{jk}(n,n+1) a_{jk}^{\text{Post}}(n)\right],$$

for every $t \in \mathbb{N}$.

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for every $t \in \mathbb{N}$.

We will derive analytic formulas for:

1. The **distribution** function $\mathbb{P}[V_t^+ < u | X_t = i]$, i.e. the probability that the prospective value does not exceed some value u, given $X_t = i$.
2. The **moments** of the random variable V_t^+, given $X_t = i$, i.e. $\mathbb{E}[(V_t^+)^p | X_t = i]$, $p \geq 1$.

Thiele's difference equation for the distribution of V_t^+

Consider the **distribution** of V_t^+, given $X_t = i$:

$$P_i(t,u) \triangleq \mathbb{P}[V_t^+ < u | X_t = i].$$

Then

$$P_i(t,u) = \mathbb{P}[V_t^+ < u, X_t = i] / \mathbb{P}[X_t = i]$$

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We know from the definition of V_t^+ that

$$V_t^+ = v_t V_{t+1}^+ + \sum_j I_j^X(t) a_j^{\text{Pre}}(t) + v_t \sum_{j,k} a_{jk}^{\text{Post}}(t) \Delta N_{jk}^X(t),$$

where $v_t = \frac{v(t+1)}{v(t)}$.

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&= \sum_{k\in\mathcal{S}} p_{ik}(t,t+1) P_k(t+1, v_t^{-1}\left(u - a_i^{\text{Pre}}(t)\right) - a_{ik}^{\text{Post}}(t)).
\end{aligned}$$

Theorem (Thiele's difference equation for distributions)

$$P_i(t,u) = \sum_{k \in \mathcal{S}} p_{ik}(t,t+1) P_k(t+1, (v_t)^{-1}(u - a_i^{Pre}(t)) - a_{ik}^{Post}(t)),$$

where $P_i(t,u)$ denotes the distribution of V_t^+ given $X_t = i$ at level $u \in \mathbb{R}$.

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Remark

Example of a terminal condition for the recursion: let $T \in \mathbb{N}$ be the maturity of the policy,

$$P_i(T,u) = \mathbb{P}[V_T^+ < u] = \begin{cases} 0, & \text{if } u \leq 0, \\ 1, & \text{if } u > 0 \end{cases}, \quad \text{if} \quad V_T^+ = 0.$$

Higher moments of V_t^+, $t \in \mathbb{N}$

Recall again the following difference equation for V_t^+,

$$V_t^+ = v_t V_{t+1}^+ + \sum_j I_j^X(t) a_j^{\text{Pre}}(t) + v_t \sum_{j,k} a_{jk}^{\text{Post}}(t) \Delta N_{jk}^X(t),$$

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Since $\sum_j I_j^X(s) = 1$ for every s we have,

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So far,

$$V_t^+ = v_t \sum_j I_j^X(t+1) V_{t+1}^+ + P_t + Q_t.$$

$$V_t^+ = v_t \sum_j I_j^X(t+1)\, V_{t+1}^+ + P_t + Q_t.$$

Hence,

$$(V_t^+)^p = \left(v_t \sum_j I_j^X(t+1)\, V_{t+1}^+ + P_t + Q_t \right)^p.$$

$$V_t^+ = v_t \sum_j l_j^X(t+1)\, V_{t+1}^+ + P_t + Q_t.$$

Hence,

$$(V_t^+)^p = \left(v_t \sum_j l_j^X(t+1)\, V_{t+1}^+ + P_t + Q_t \right)^p.$$

Remark (Newton's binomial formula)

Recall

$$(x_1 + \cdots + x_m)^n = \sum_{\substack{k_1,\dots,k_m,=0,\dots,n \\ k_1+\cdots+k_m=n}} \binom{n}{k_1,\dots,k_m} x_1^{k_1} \cdots x_m^{k_m},$$

where

$$\binom{n}{k_1,\dots,k_m} = \frac{n!}{k_1!\cdots k_n!}.$$

$$V_t^+ = v_t \sum_j I_j^X(t+1)\, V_{t+1}^+ + P_t + Q_t.$$

Hence,

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Apply the above formula with $m = 3$ to the expression $(V_t^+)^p$ above.

$$
\begin{aligned}
(V_t^+)^p &= \left(v_t \sum_j I_j^X(t+1) V_{t+1}^+ + P_t + Q_t \right)^p \\
&= \sum_{\substack{k_1,k_2,k_3,=0,\dots,p \\ k_1+k_2+k_3=p}} \binom{p}{k_1, k_2, k_3} \left(v_t \sum_j I_j^X(t+1) V_{t+1}^+ \right)^{k_1} (P_t)^{k_2} (Q_t)^{k_3} \\
&= \sum_{\substack{k_1,k_2,k_3,=0,\dots,p \\ k_1+k_2+k_3=p}} \binom{p}{k_1, k_2, k_3} v_t^{k_1} \sum_j I_j^X(t+1) (V_{t+1}^+)^{k_1} (P_t)^{k_2} (Q_t)^{k_3}
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\end{aligned}
$$

Next step is to apply $\mathbb{E}[\cdot | X_t = i]$.

$$\mathbb{E}[(V_t^+)^p|X_t = i] = \sum_{\substack{k_1,k_2,k_3,=0,\ldots,p \\ k_1+k_2+k_3=p}} \binom{p}{k_1,k_2,k_3} v_t^{k_1} \sum_j \mathbb{E}[I_j^X(t+1)(V_{t+1}^+)^{k_1}(P_t)^{k_2}(Q_t)^{k_3}|X_t = i]$$

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Observe that

$$\mathbb{E}[I_j^X(t+1)(V_{t+1}^+)^{k_1}(P_t)^{k_2}(Q_t)^{k_3}|X_t = i] = \mathbb{E}[I_j^X(t+1)(V_{t+1}^+)^{k_1}(a_i^{\text{Pre}}(t))^{k_2}(v_t a_{ij}^{\text{Post}}(t))^{k_3}|X_t = i]$$

$$\mathbb{E}[(V_t^+)^p|X_t = i] = \sum_{\substack{k_1,k_2,k_3,=0,\dots,p \\ k_1+k_2+k_3=p}} \binom{p}{k_1,k_2,k_3} v_t^{k_1} \sum_j \mathbb{E}[I_j^X(t+1)(V_{t+1}^+)^{k_1}(P_t)^{k_2}(Q_t)^{k_3}|X_t = i]$$

Observe that

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The reason is that, given $X_t = i$, $P_t = \sum_j I_j^X(t) a_j^{\text{Pre}}(t) = a_i^{\text{Pre}}(t)$. Similarly, $\Delta N_{jk}^x(t) = N_{jk}^X(t+1) - N_{jk}^X(t)$ given $X_t = i$ equals $\Delta N_{ik}^x(t)$. Hence, $I_j^X(t+1)\Delta N_{jk}^X(t)$ given $X_t = i$ equals $N_{ij}^X(t)$. Thus, given $X_t = i$

$$I_j^X(t+1)(Q_t)^{k_3} = I_j^X(t+1)\left(v_t \sum_{j,k} a_{jk}^{\text{Post}}(t)\Delta N_{jk}^X(t)\right)^{k_3} = I_j^x(t+1)v_t^{k_3}(a_{ij}^{\text{Post}}(t))^{k_3}.$$

$$\mathbb{E}[(V_t^+)^p | X_t = i] = \sum_{\substack{k_1,k_2,k_3,=0,\dots,p \\ k_1+k_2+k_3=p}} \binom{p}{k_1, k_2, k_3} v_t^{k_1} \sum_j \mathbb{E}[I_j^X(t+1)(V_{t+1}^+)^{k_1}(P_t)^{k_2}(Q_t)^{k_3} | X_t = i]$$

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$$I_j^X(t+1)(Q_t)^{k_3} = I_j^X(t+1)\left(v_t \sum_{j,k} a_{jk}^{\text{Post}}(t)\Delta N_{jk}^X(t)\right)^{k_3} = I_j^x(t+1) v_t^{k_3} (a_{ij}^{\text{Post}}(t))^{k_3}.$$

$$\mathbb{E}[(V_t^+)^p | X_t = i] =$$

$$= \sum_{\substack{k_1,k_2,k_3,=0,\dots,p \\ k_1+k_2+k_3=p}} \binom{p}{k_1, k_2, k_3} v_t^{k_1+k_3} \sum_j \mathbb{E}[I_j^X(t+1)(V_{t+1}^+)^{k_1}(a_i^{\text{Pre}}(t))^{k_2}(a_{ij}^{\text{Post}}(t))^{k_3} | X_t = i]$$

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Now,

$$\mathbb{E}\left[I_j^X(t+1)\left(V_{t+1}^+\right)^{k_1}\middle|X_t=i\right]=\frac{1}{\mathbb{P}[X_t=i]}\mathbb{E}\left[I_j^X(t+1)\left(V_{t+1}^+\right)^{k_1}I_i^X(t)\right]$$

So far,

$$\mathbb{E}[(V_t^+)^p | X_t = i] =$$

$$= \sum_{\substack{k_1, k_2, k_3, = 0, \dots, p \\ k_1 + k_2 + k_3 = p}} \binom{p}{k_1, k_2, k_3} v_t^{k_1 + k_3} (a_i^{\text{Pre}}(t))^{k_2} (a_{ij}^{\text{Post}}(t))^{k_3} \sum_j \mathbb{E}[I_j^X(t+1)(V_{t+1}^+)^{k_1} | X_t = i].$$

Now,

$$\begin{aligned} \mathbb{E}\left[I_j^X(t+1) \left(V_{t+1}^+\right)^{k_1} \middle| X_t = i \right] &= \frac{1}{\mathbb{P}[X_t = i]} \mathbb{E}\left[I_j^X(t+1) \left(V_{t+1}^+\right)^{k_1} I_i^X(t) \right] \\ &= \frac{\mathbb{P}[X_{t+1} = j]}{\mathbb{P}[X_t = i]} \frac{1}{\mathbb{P}[X_{t+1} = j]} \mathbb{E}\left[I_j^X(t+1) \left(V_{t+1}^+\right)^{k_1} I_i^X(t) \right] \end{aligned}$$

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Now,

$$\begin{aligned}
\mathbb{E}\left[I_j^X(t+1)\left(V_{t+1}^+\right)^{k_1} \middle| X_t = i\right] &= \frac{1}{\mathbb{P}[X_t = i]} \mathbb{E}\left[I_j^X(t+1)\left(V_{t+1}^+\right)^{k_1} I_i^X(t)\right] \\
&= \frac{\mathbb{P}[X_{t+1} = j]}{\mathbb{P}[X_t = i]} \frac{1}{\mathbb{P}[X_{t+1} = j]} \mathbb{E}\left[I_j^X(t+1)\left(V_{t+1}^+\right)^{k_1} I_i^X(t)\right] \\
&= \frac{\mathbb{P}[X_{t+1} = j]}{\mathbb{P}[X_t = i]} \mathbb{E}\left[\left(V_{t+1}^+\right)^{k_1} I_i^X(t) | X_{t+1} = j\right]
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&= \frac{\mathbb{P}[X_{t+1} = j]}{\mathbb{P}[X_t = i]}\mathbb{E}\left[\left(V_{t+1}^+\right)^{k_1} I_i^X(t)|X_{t+1} = j\right] \\
&= \mathbb{P}[X_{t+1} = j|X_t = i]\mathbb{E}\left[\left(V_{t+1}^+\right)^{k_1}|X_{t+1} = j\right]
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So far,

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&= \frac{\mathbb{P}[X_{t+1} = j]}{\mathbb{P}[X_t = i]}\mathbb{E}\left[\left(V_{t+1}^+\right)^{k_1} I_i^X(t)|X_{t+1} = j\right] \\
&= \mathbb{P}[X_{t+1} = j|X_t = i]\mathbb{E}\left[\left(V_{t+1}^+\right)^{k_1}|X_{t+1} = j\right] \\
&= p_{ij}(t, t+1)\mathbb{E}\left[\left(V_{t+1}^+\right)^{k_1}|X_{t+1} = j\right].
\end{aligned}$$

Theorem (Moments of the stochastic prospective value)

Let $p \geq 1$ integer and V_t^+, $t \in \mathbb{N}$ be the stochastic prospective value. Let

$$M_i^p(t) \triangleq \mathbb{E}[(V_t^+)^p | X_t = i]$$

denote the p-th moment of V_t^+ given $X_t = i$. Then $M_i^p(t)$ satisfies

$$M_i^p(t) = \sum_{j \in S} p_{ij}(t, t+1) \sum_{\substack{k_1, k_2, k_3, =0, \ldots, p \\ k_1+k_2+k_3=p}} \binom{p}{k_1, k_2, k_3} (v_t)^{k_1+k_3} (a_i^{Pre}(t))^{k_2} (a_{ij}^{Post}(t))^{k_3} M_j^{k_1}(t+1).$$

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Corollary

If $a_i^{Pre}(t) = 0$ *for all* i, *then the equation reduces to:*

$$M_i^p(t) = v_t^p \sum_j p_{ij}(t, t+1) \sum_{k=0}^{p} \binom{p}{k} (a_{ij}^{Post}(t))^{p-k} M_j^k(t+1).$$

Examples

Let us look at an endowment insurance with benefit E. We wish to compute

$$P_*(t,u) = \mathbb{P}[V_t^+ < u | X_t = *].$$

Observe that

$$P_*(T,u) = \mathbb{I}(u > E) \quad P_\dagger(T,u) = \mathbb{I}(u > 0).$$

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The general formula is:

$$P_i(t,u) = \sum_{k \in \mathcal{S}} p_{ik}(t,t+1) P_k(t+1, (v_t)^{-1}(u - a_i^{\text{Pre}}(t)) - a_{ik}^{\text{Post}}(t)).$$

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Thus,

$$P_*(t,u) = p_{**}(t,t+1) P_*(t+1, v_t^{-1} u) + p_{*\dagger}(t,t+1) P_\dagger(t+1, v_t^{-1} u).$$

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Iterating one can show that $P_\dagger(t,u) = \mathbb{I}(u > 0)$ for all $t = 0, 1, \ldots, T$ and

$$P_*(t,u) = p_{**}(t,T) \mathbb{I}\left(u > \frac{v(T)}{v(t)} E\right) + \sum_{n=t}^{T-1} p_{**}(t,n) p_{*\dagger}(n,n+1) \mathbb{I}(u > 0).$$

On the other hand observe that

$$\begin{aligned}
\sum_{n=t}^{T-1} p_{**}(t,n)p_{*\dagger}(n,n+1) &= \sum_{n=t}^{T-1} p_{**}(t,n)\left(1 - p_{**}(n,n+1)\right) \\
&= \sum_{n=t}^{T-1} p_{**}(t,n) - \sum_{n=t}^{T-1} p_{**}(t,n+1) \\
&= \sum_{n=t}^{T-1} p_{**}(t,n) - \sum_{n=t}^{T-1} p_{**}(t,n+1) \\
&= 1 - p_{**}(t,T).
\end{aligned}$$

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Hence,

$$P_*(t,u) = p_{**}(t,T)\mathbb{I}\left(u > \frac{v(T)}{v(t)}E\right) + \left(1-p_{**}(t,T)\right)\mathbb{I}(u>0).$$

$P_*(t,u)$ is indeed a distribution function that looks like

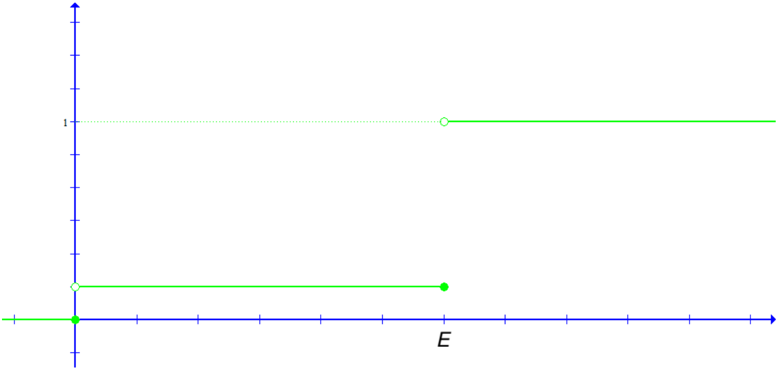

Figure: Shape of the distribution function of V_t^+.

$$P_*(t,u) = p_{**}(t,T)\mathbb{I}\left(u > \frac{v(T)}{v(t)}E\right) + (1 - p_{**}(t,T))\,\mathbb{I}(u > 0).$$

To see that observe that $\lim_{u \to \infty} P_*(t,u) = 0$, $P_*(t,\cdot)$ is increasing and

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The distribution at $t = 0$ is given by

$$P_*(0,u) = p_{**}(0,T)\mathbb{I}\,(u > v(T)E) + (1 - p_{**}(0,T))\,\mathbb{I}(u > 0)$$

which makes sense.

For the term insurance $a_i^{\text{Pre}}(t) = 0$, so we can use:

$$M_i^p(t) = v_t^p \sum_j p_{ij}(t, t+1) \sum_{k=0}^{p} \binom{p}{k} (a_{ij}^{\text{Post}}(t))^{p-k} M_j^k(t+1).$$

from slide 14.

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from slide 14.
Using $\mathcal{S} = \{*, \dagger\}$ and $a_{*\dagger}^{\text{Post}}(t) = B$ for $t = 0, \ldots, T-1$ and denoting $M_i^p(t) = \mathbb{E}[(V_t^+)^p | X_t = i]$, we get

$$M_*^2(t) = (v_t)^2 \left[p_{**}(t, t+1) M_*^2(t+1) + p_{*\dagger}(t, t+1) B^2 \right],$$

where we used that $M_\dagger^k(t) = 0$ for $k = 1, 2$ and $a_{**}^{\text{Post}}(t) = 0$.

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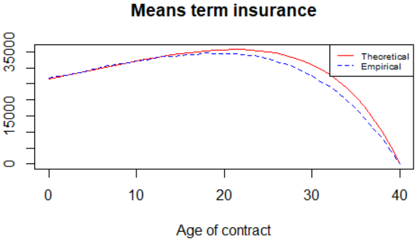
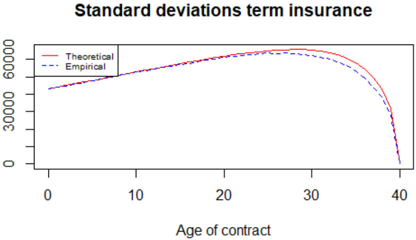
where we used that $M_\dagger^k(t) = 0$ for $k = 1, 2$ and $a_{**}^{\text{Post}}(t) = 0$.
We compute $\mathbb{V}[V_t^+ | X_t = *] = M_*^2(t) - (V_*^+(t))^2$ with parameters: $B = 200\,000$, $r = 3\%$, age $x = 30$, $T = 40$ using both a Monte-Carlo method with $N = 10\,000$ iterations and the theoretical formula above.

The (conditional) means and standard deviations we obtained are:



(a) Means



(b) Standard deviations

Figure: Term insurance with death benefit $B = 200\,000$, $r = 3\%$, $x = 30$, $T = 40$. Theoretical vs. Empirical quantities with $N = 10\,000$ simulations.

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Life Insurance and Finance

Lecture 10: Distributional properties of the stochastic prospective value