UiO : Department of Mathematics
University of Oslo

Life Insurance and Finance

Lecture 11: Stochastic integration

David R. Banos

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Preliminaries

Recall that we already defined stochastic integrals of the form

$$(Y \cdot A)_t := \int_0^t Y(s,\omega)\, d\underbrace{A(s,\omega)}_{\text{integrator}} \tag{1}$$

where $A = \{A(t,\omega), t \geq 0, \omega \in \Omega\}$ (cash flow) is a **stochastic process** on a given probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with paths which are right-continuous with existing left limits and of bounded variation.

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Problem: Can we define (1) for more general integrators $X(s,\omega)$ as e.g.

$$X(s,\omega) = A(s,\omega) + B(s,\omega),$$

where $B(s,\omega)$ is a Brownian motion or more generally, for processes $X_s = X(s,\omega)$ of the form

$$X_s = X_0 + A_s + M_s,$$

where M is a **martingale** or more generally a **local martingale**?

In order to define local martingales we need two concepts:

Definition (Usual hypotheses)

A filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$ (e.g. market information flow) is said to satisfy the ***usual hypotheses*** if

(i) $\mathcal{F}_0$ contains all $N \in \mathcal{A}$ with $\mathbb{P}(N) = 0$. Meaning that market traders are aware of the possible occurrence of extremely rare market events N.

(ii) $\mathcal{F}$ is right-continuous, that is

$$\mathcal{F}_t = \bigcap_{u>t} \mathcal{F}_u, \quad t \geq 0,$$

meaning that the market information flow does not "jump" when a market analyst looks back into history.

Remark

Implication of this: $X = \{X_t, t \geq 0\}$ adapted to $\mathcal{F}$ (i.e. X_t is $\mathcal{F}_t$-measurable for all $t \geq 0$) and $Y = \{Y_t, t \geq 0\}$ a stochastic process such that $X_t = Y_t$ with probability one, then Y is also $\mathcal{F}$ adapted.

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Definition (Stopping time)

A random variable τ with values in $[0, \infty]$ is a stopping time if

$$\{\tau \leq t\} \in \mathcal{F}_t$$

for all $t \geq 0$.

Example (Stopping time)

S_t, $t \geq 0$ (right-continuous) stock price process. Define

$$\tau := \inf \{t \geq 0 : S_t = 100\$\}$$

first time that S_t reaches 100$. Here, $\inf \emptyset = \infty$ by convention.

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Example (Not stopping time)

S_t, $t \geq 0$ (right-continuous) stock price process. Define

$$\tau := \text{argmax}\{S_t, t \in [0, T]\}$$

the maximum value of the stock on the time frame $[0, T]$. The event $\{\tau > t\}$ is not in our knowledge in $\mathcal{F}_t$.

Definition (Local martingale)

An adapted process $M = \{M_t, t \geq 0\}$ with right-continuous paths and existing left limits (i.e. an adapted càdlàg process) is a local martingale if there are increasing stopping times τ_n, $n \geq 1$ with $\tau_n \to \infty$ as $n \to \infty$ with probability one such that

(i) the stopped process (on $\{\tau_n > 0\}$)

$$M_t^{\tau_n} \triangleq M_{t \wedge \tau_n} \mathbf{1}_{\{\tau_n > 0\}}$$

is a martingale for each n, that is

$$\mathbb{E}[M_t^{\tau_n} | \mathcal{F}_s] = M_s^{\tau_n}, \quad t \geq s$$

for each $n \geq 1$, and

Definition (Local martingale)

(ii)

$$\sup_{t\geq 0} \mathbb{E}\left[|M_t^{\tau_n}|\mathbb{I}_{\{|M_t^{\tau_n}|\geq m\}}\right] \xrightarrow{m\to\infty} 0$$

for each $n \geq 1$ (uniform integrability).

Comment:

- Every **martingale** is a **local martingale**.
- Every **local martingale** that is **bounded from below** is a **supermartingale**, and every **local martingale** that is **bounded from above** is a **submartingale**. Hence, every bounded **local martingale** is a **martingale**. However, in general a local martingale is **not** a martingale, because its expectation can be distorted by large values of small probability.

Example

An example of a local martingale which is not a martingale is: $X_t = \log|W_t - 1|$, $t \geq 0$, where W is a complex-valued Brownian motion. Also, the solution of the SDE $dX_t = X_t^\alpha dW_t$ where $\alpha > 1$ is a local martingale.

Also, if you consider a geometric Brownian motion, i.e. $X_t = X_0 e^{-\frac{1}{2}\sigma^2 t + \sigma W_t}$, then X is a martingale for $t \geq 0$, now define $Y_t = X_{f(t)}$ where $f(t) = \tan((\pi/2)t)$, then Y is a local martingale on $[0, 1]$ but not a martingale. Any stochastic integral $\int_0^t f(s) dW_s$ where $f = f(s, \omega)$ is such that $\int_0^T |f(s)|^2 dt < \infty$ $P - a.s.$ is a local martingale, not necessarily a martingale. If, in addition, $\int_0^T \mathbb{E}[|f(s)|^2] dt < \infty$ then it is a martingale.

Definition (Semimartingale)

An $\mathcal{F}$-adapted càdlàg process X is a semimartingale if

$$X_t = X_0 + A_t + M_t, \quad t \geq 0, \tag{2}$$

where A and M are càdlàg adapted processes such that A is of bounded variation (with probability one) and M is a local martingale.

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Remark

If $A_0 = M_0 = 0$ and A is continuous in t then the decomposition of X in (2) is unique, i.e.

$$X_t = 0, \quad t \geq 0 \iff A_t = 0, M_t = 0, t \geq 0.$$

Local martingales in...

... stochastic analysis: The Itô integral is classically only constructed for processes ξ_s that are in $L^2_a([0,T])$ and the Itô integral is a **martingale**. If we want to extend the construction more generally to ξ which are not square-integrable, we can, but then the integral is a **local-martingale**.

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... finance: When pricing contracts, we use a pricing formula based on martingality:

$$\widetilde{V}_t = \mathbb{E}_{\mathbb{Q}}[\widetilde{V}_T|\mathcal{F}_t],$$

where V_t is the value of a hedging portfolio at time t and $\mathbb{Q}$ is a so-called pricing measure. We want to hedge some r.v. $V_T = H$. Even if the (discounted) underlying tradable assets are martingales under $\mathbb{Q}$, the associated portfolio V need not be a martingale, but merely a local martingale. Then the price may be wrong. Local martingales are hence used for explaining **financial bubbles**.

Motivation

Motivation of how to define a stochastic integral:

$$\int_0^t Y_s dX_s. \tag{3}$$

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Recall the **Riemann integral**:

$$\int_0^T f(s)ds = \lim_{|\pi|\to 0} \sum_{i=1}^{n-1} f(\xi_i)(t_{i+1} - t_i), \tag{4}$$

for partitions $\pi = \{t_i\}_{i=0}^n$ of $[0, T]$ with mesh $|\pi| = \max_{i=1,\dots,n-1} |t_{i+1} - t_i| \to 0$, $0 = t_0 < t_1 < \cdots < t_n = T$, $\xi_i \in [t_i, t_{i+1}]$.

If f is a step function, i.e.

$$f(s) = \sum_{i=0}^{n-1} \underbrace{a_i}_{\in \mathbb{R}} \mathbb{I}_{(t_i, t_{i+1}]}(s), \quad s \in (0, T]$$

for $0 = t_0 < t_1 < \cdots < t_n = T$ then

$$\int_0^T f(s)ds = \sum_{i=0}^{n-1} a_i(t_{i+1} - t_i) = \sum_{i=0}^{n-1} a_i \Delta t_i, \tag{5}$$

where $\Delta t_i = t_{i+1} - t_i$, $i = 0, \ldots, n-1$.

Although it is tedious, one can compute integrals via its definition:

Example

Let $f(x) = x^2$ and $\pi = \{t_i\}_{i=0}^{n}$, $t_i = \frac{i}{n}$, $i = 0, \ldots, n$ then

$$\int_0^1 f(x)dx = \lim_n \sum_{i=1}^{n-1} f(t_i)(t_{i+1} - t_i) = \lim_n \sum_{i=1}^{n-1} t_i^2 \frac{1}{n}$$
$$= \lim_n \frac{1}{n} \sum_{i=1}^{n-1} \left(\frac{i}{n}\right)^2 = \lim_n \frac{1}{n^3} \sum_{i=1}^{n-1} i^2$$
$$= \lim_n \frac{1}{n^3} \frac{(n-1)n(2n-1)}{6} = \frac{1}{3}.$$

Construction via simple predictable processes

It is reasonable to define the stochastic integral (3) of **stochastic step functions** Y_t, $t \geq 0$ similarly to (5), that is if

$$Y_s = H_0 \mathbf{1}_{\{0\}}(s) + \sum_{i=1}^{n} H_i \mathbb{I}_{(T_i, T_{i+1}]}(s), \tag{6}$$

where $H_0, H_1, \ldots, H_n$ and $T_1, \ldots, T_{n+1}$ are **random variables** instead of real numbers t_i and a_i as before. Then...

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$$\int_0^t Y_s dX_s := H_0 X_0 + \sum_{i=1}^{n} H_i \underbrace{(X_{t\wedge T_{i+1}} - X_{t\wedge T_i})}_{\underbrace{X_{T_{i+1}} - X_{T_i}}_{\Delta X_{T_i}} \text{ if } t \geq T_{i+1}} \tag{7}$$

for $0 = T_1 \leq T_2 \leq \cdots \leq T_{n+1} < \infty$.

In applications to finance and insurance, however, Y is typically given by a **hedging strategy** with respect to a **financial instrument** X or a discount factor based on stochastic interest rates integrated against a **stochastic cash flow** X. In order to rule out portfolio strategies Y of an insider, it is assumed that Y is based on **market information** up to time t (i.e. $\mathcal{F}_t$), that is Y is adapted to $\mathcal{F}$.

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if $0 = T_0 \leq T_1 \leq \cdots \leq T_{n+1} < \infty$ are **stopping times** and H_i is based on market information up to the random time T_i, that is a random variable with respect to $\mathcal{F}_{T_i}$ defined hereunder.

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Definition (Stopping time σ-algebra $\mathcal{F}_\tau$)

For a stopping time τ we define the **stopping σ-algebra** as

$$\mathcal{F}_\tau := \{A \in \mathcal{A} : A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}.$$

Definition (Stochastic integral of stochastic step functions)

Let $Y = \{Y_t, t \geq 0\}$ be given by (6) with $0 = T_0 \leq T_1 \leq \cdots \leq T_{n+1} < \infty$ and H_i are random variables on $(\Omega, \mathcal{F}_{T_i}, P)$ such that $|H_i| < \infty$ with probability one, $i = 0, \ldots, n$. Then the stochastic integral of Y with respect to X is defined as in (7) ,

$$\int_0^t Y_s dX_s := H_0 X_0 + \sum_{i=1}^{n} H_i (X_{t\wedge T_{i+1}} - X_{t\wedge T_i}) \tag{8}$$

and denoted by

$$\int_0^t Y_s dX_s \quad \text{or} \quad (Y \cdot X)_t. \tag{9}$$

Remark

(i) Processes of the form (6) are also called **simple predictable** and the collection of those processes is usually denoted by S or $\mathcal{S}$.

(ii) The definition of (9) is **independent** of the specific representation in (6).

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(ii) The definition of (9) is **independent** of the specific representation in (6).

In view of applications we find that the class S of integrand processes is rather small.

The natural approach is to define

$$\int_0^t Y_s dX_s$$

for more general integrand processes Y by approximation of Y through stochastic step functions $Y^{(n)} \in S$.

In general, this approximation can **not** be as in (4) in the sense of convergence in $\mathbb{R}$, since e.g. the Brownian motion $X_s = B_s$ is not of bounded variation.
However, this approximation can be in the sense of convergence in **probability** (or L^2) or more precisely, in the following sense:

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However, this approximation can be in the sense of convergence in **probability** (or L^2) or more precisely, in the following sense:

Let Y and Z be processes. We define the distance between Y and Z as

$$d(Y,Z) := \sum_{n\geq 1} \frac{1}{2^n} \mathbb{E}\left[1 \wedge \sup_{0\leq s\leq n} |Y_s - Z_s|\right],$$

where $x \wedge y := \min\{x,y\}$. Then

$$Y^{(n)} \xrightarrow{n\to\infty} Y$$

with respect to d, if, and only if

$$d(Y^{(n)}, Y) \xrightarrow{n\to\infty} 0. \tag{10}$$

Remark

The convergence in (10) is equivalent to: for all sequences there exists a subsequence $\{n_k\}_{k\geq 0} \subset \mathbb{N}$ of the latter such that

$$Y_t^{(n_k)} \xrightarrow{k\to\infty} Y_t \tag{11}$$

with probability one **uniformly** in t (on compact intervals).

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with probability one **uniformly** in t (on compact intervals).

In the following denote by $\mathbb{D}$ the collection of adapted processes with **right-continuous paths** and **existing left-limits** (càdlag) and by $\mathbb{L}$ the class of adapted processes with **left-continuous paths** and **existing right-limits** (làdcàg). Obviously, $\mathbb{L} \supset S$.
The following lemma is the starting point for the construction of stochastic integrals of $Y \in \mathbb{L}$.

Lemma

Each $Y \in \mathbb{L}$ *can be approximated by* $\{Y^{(m)}\}_{m\geq 1} \subset S$ *in the sense of* (10), *or equivalently of* (11).

Definition (Stochastic integral for $\{Y_t\}_{t\geq 0} \in \mathbb{L}$)

Let $\{Y_t^{(m)}\}_{t\geq 0} \subset S$ be q sequence of simple predictable stochastic processes as defined in (6) and $\{Y_t\}_{t\geq 0} \in \mathbb{L}$ such that $\{Y_t^{(m)}\}_{m\geq 1}$ approximates Y in the sense of (10), or equivalently of (11). The then unique process $Z \in \mathbb{D}$ with

$$d\left(\int_0^{\cdot} Y_s^{(m)} dX_s\,,\, Z\right) \xrightarrow{m\to\infty} 0$$

is called stochastic integral of Y with respect to X and denoted by

$$Z_t = \int_0^t Y_s dX_s, \quad \text{or} \quad Z_t = (Y \cdot X)_t, \quad t \geq 0.$$

Some properties of the stochastic integral

(i)

$$\int_0^t Y_s d(\alpha X_s + \beta V_s) = \alpha \int_0^t Y_s dX_s + \beta \int_0^t Y_s dV_s.$$

(ii) $V_t := \int_0^t Y_s dX_s$, $\{R_t\}_{t\geq 0} \in \mathbb{L}$ then V is a semimartingale and

$$\int_0^t R_s dV_s = \int_0^t R_s Y_s dX_s.$$

(iii) X a local martingale. Then

$$M_t := \int_0^t Y_s dX_s, \quad t \geq 0$$

is a local martingale, too.

(iv) τ stopping time, then

$$\int_0^{t\wedge\tau} Y_s dX_s = \int_0^t Y_s \mathbf{1}_{[0,\tau]}(s) dX_s.$$

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(v) $\Delta f(t) := f(t) - f(t^-)$ and $V_t := \int_0^t Y_s dX_s$, then

$$\Delta V_t = Y_t \Delta X_t.$$

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(v) $\Delta f(t) := f(t) - f(t^-)$ and $V_t := \int_0^t Y_s dX_s$, then

$$\Delta V_t = Y_t \Delta X_t.$$

(vi) **Itô isometry**: $X_t = B_t$, $t \geq 0$, Brownian motion. Then $\mathbb{E}\left[\int_0^t Y_s dB_s\right] = 0$ and

$$\mathbb{V}ar\left[\int_0^t Y_s dB_s\right] = \mathbb{E}\left[\left(\int_0^t Y_s dB_s\right)^2\right] = \mathbb{E}\left[\int_0^t Y_s^2 ds\right].$$

The above fact comes from

$$\left[\int_0^\cdot Y_s dB_s, \int_0^\cdot Y_s dB_s\right]_t = \int_0^t Y_s^2 ds.$$

Quadratic covariation

Definition (Quadratic covariation)

Then the covariation of two processes $X = \{X_t, t \in [0, T]\}$ and $Y = \{Y_t, t \in [0, T]\}$ is defined as

$$[X, Y]_t = \lim_{|\pi| \to 0} \sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}})$$

whenever the limit exists, understood in the probability sense.

Definition (Quadratic variation)

The quadratic variation of a process $X = \{X_t, t \in [0, T]\}$ is defined as

$$[X]_t = [X, X]_t,$$

whenever the limit exists, in probability.

Definition (Quadratic variation process)

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Definition (Quadratic covariation process)

Let $V = \{V_t\}_{t\geq 0}$ and $Z = \{Z_t\}_{t\geq 0}$ be semimartingales. Then the quadratic covariation process of V and Z is defined as the process

$$[V, Z]_t := V_t Z_t - \int_0^t V_{s-} dZ_s - \int_0^t Z_{s-} dV_s, \quad t \geq 0. \tag{12}$$

Some properties of the quadratic variation

(i) $[X,X]_\cdot \in \mathbb{D}$ and increasing.

(ii) $[X,X]_0 = X_0^2$ and $\Delta[X,X]_t = (\Delta X_t)^2$.

(iii) $0 = \tau_0^n \leq \tau_1^n \leq \cdots \leq \tau_{k_n}^n$, $n \geq 1$ stopping times with

$$\sup_k \tau_k^n \xrightarrow{n\to\infty} \infty \text{ and } \sup_k |\tau_{k+1}^n - \tau_k^n| \xrightarrow{n\to\infty} 0,$$

i.e. a random partition with vanishing mesh. Then

$$X_0^2 + \sum_{i\geq 1}\left(X_{t\wedge\tau_{i+1}^n} - X_{t\wedge\tau_i^n}\right)^2 \xrightarrow{n\to\infty} [X,X]_t$$

in the sense of (10), or equivalently of (11).

(iv) $\{X_t\}_{t\geq 0} \in \mathbb{D}$, X of bounded variation. Then

$$[X,X]_t = X_0^2 + \sum_{0<s\leq t} (\Delta X_s)^2.$$

Alternative way to construct stochastic integrals via an isometry

The following theorem provides an alternative way of defining the stochastic integral $Z_t = (Y \cdot X)_t$ of Y w.r.t. a semimartingale X.

Theorem (Stochastic integral)

Let X and Y be two adapted square-integrable martingales, then $(Y \cdot X)$ is the unique element such that $[(Y \cdot X), N] = Y \cdot [X, N]$, for all N. Then the map $Y \mapsto (Y \cdot X)$ is an isometry on right spaces.

Remark (Isometry property)

$$[K \cdot X, H \cdot Y]_t = \int_0^t K_s H_s d[X, Y]_s.$$

In particular,

$$[K \cdot X, K \cdot X]_t = \int_0^t |K_s|^2 d[X, X]_s.$$

Itô formula (chain rule)

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$$\int_0^t Y_s dX_s \, ?$$

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1. Method: Direct use of the definition of stochastic integral. But this is, as mentioned, in general rather tedious.

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1. Method: Direct use of the definition of stochastic integral. But this is, as mentioned, in general rather tedious.
2. Method: **Itô's formula**: it can be considered a chain rule for semimartingales.

Theorem (Itô's formula for semimartingales)

Let X be a semimartingale. Denote by $[X,X]_t^c$ the continuous part of $[X,X]_t$, that is

$$\underbrace{[X,X]_t^c}_{\text{of B.V.}} = [X,X]_t - X_0^2 - \underbrace{\sum_{0<s\leq t}}_{\substack{\text{abs. convergent} \\ \text{countable sum}}} (\Delta X_s)^2.$$

Theorem (Itô's formula for semimartingales)

Let f *be a function in* $C^2(\mathbb{R})$ *(i.e. twice continuously differentiable). Then the process* $Z_t := f(X_t)$ *is a semimartingale again and*

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_{s-})dX_s + \frac{1}{2}\int_0^t f''(X_{s-})d[X,X]^c_s \\ &\quad + \sum_{0<s\leq t}\left[f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s\right] \\ &= f(X_0) + \int_0^t f'(X_{s-})dX_s + \frac{1}{2}\int_0^t f''(X_{s-})d[X,X]_s \\ &\quad + \sum_{0<s\leq t}\left[f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s - \frac{1}{2}f''(X_{s-})(\Delta X_s)^2\right]. \end{aligned} \tag{13}$$

Proof.

The proof is based on Taylor's formula. ∎

Remark

Convention in (13):

$$f'(X_{0-}) = 0 \text{ and } f''(X_{0-}) = 0. \tag{14}$$

Remark

In the lecture notes you can find a multidimensional version of the Itô formula.

Example

$B = \{B_t\}_{t\in[0,T]}$, $T > 0$ fixed, a Brownian motion. What is $\int_0^t B_s dB_s$?

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$B = \{B_t\}_{t\in[0,T]}$, $T > 0$ fixed, a Brownian motion. What is $\int_0^t B_s dB_s$?
Choose $X_t = B_t$ and $f(x) = x^2$ in the theorem for Itô's formula. B continuous $\Rightarrow \Delta X_s = \Delta B_s = 0$ and $f(X_s) - f(X_{s-}) = 0$, hence the countable sum in (13) is zero. Thus,

$$\underbrace{f(X_t)}_{=B_t^2} = \underbrace{f(X_0)}_{=0} + \int_0^t \underbrace{f'(X_{s-})}_{2B_s^- = 2B_s} d \underbrace{X_s}_{B_s} + \frac{1}{2}\int_0^t \underbrace{f''(X_{s-})}_{=2} d \underbrace{[X,X]_s^c}_{=s}.$$

Example

$B = \{B_t\}_{t\in[0,T]}$, $T > 0$ fixed, a Brownian motion. What is $\int_0^t B_s dB_s$? Choose $X_t = B_t$ and $f(x) = x^2$ in the theorem for Itô's formula. B continuous $\Rightarrow \Delta X_s = \Delta B_s = 0$ and $f(X_s) - f(X_{s-}) = 0$, hence the countable sum in (13) is zero. Thus,

$$\underbrace{f(X_t)}_{=B_t^2} = \underbrace{f(X_0)}_{=0} + \int_0^t \underbrace{f'(X_{s-})}_{2B_s^- = 2B_s} d\underbrace{X_s}_{B_s} + \frac{1}{2}\int_0^t \underbrace{f''(X_{s-})}_{=2} d\underbrace{[X,X]_s^c}_{=s}.$$

As a result,

$$\int_0^t B_s dB_s = \frac{1}{2}B_t^2 - \frac{1}{2}t.$$

Itô processes and stochastic differential equations (SDEs)

Although we have (very quickly) seen a theory for integration w.r.t. very general semimartingale processes (even discontinuous), we will focus mostly on the case $X_s = B_s$ and the integral is the classical Itô integral. Hence, we know how to construct integrals with ds (Lebesgue) and with dB_s (Itô). So, given two adapted processes u and v such that $\int_0^t \mathbb{E}[|u_s|]ds < \infty$ and $\int_0^t \mathbb{E}[|v_s|^2]ds < \infty$ we can look at processes of the form

$$X_t = X_0 + \int_0^t u_s ds + \int_0^t v_s dB_s.$$

Such processes are known as **Itô processes.**

If we choose u and v in the previous slide as two deterministic functions on $b, \sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ and define $u_s = b(s, X_s)$ and $v_s = \sigma(s, X_s)$ then we are looking at processes like

$$X_t = X_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s. \tag{15}$$

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$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s. \tag{15}$$

Written in differential form would be:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad t \in [0, T].$$

The above mathematical expression is known as a stochastic differential equation. It describes the dynamics of X (in differential form) and it has no other meaning than (15)

UiO : Department of Mathematics
University of Oslo

David R. Banos

Life Insurance and Finance

Lecture 11: Stochastic integration