UiO : Department of Mathematics
University of Oslo

Life Insurance and Finance

Lecture 2: Numerical methods for ODEs

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STK4500

1 ODEs

2 Euler's method

ODEs

Let $d \geq 0$ and $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^d$ some fixed starting time and position.
Let $f : [t_0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ be a time dependent vector field.
A first-order differential equation is a Cauchy problem, also called initial value problem (IVP) of the form

$$x'(t) = f(t, x(t)), \quad t \in [t_0, \infty), \quad x(t_0) = x_0 \in \mathbb{R}^d. \tag{1}$$

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If $f(t, y) = y$ then $f(t, x(t)) = x(t)$ and hence $x'(t) = x(t)$. A function who derivative is itself is of the form $x(t) = Ce^t$. If $x(t_0) = x_0$ then $x(t) = x_0 e^{t-t_0}$, $t \in [t_0, \infty)$.

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For example, the second-order equation $x''(t) = f(t, x(t))$ can be reduced by defining $y(t) \triangleq x'(t)$.

Then $y'(t) = x''(t) = f(t, x(t))$ and we have the equation

$$z'(t) = F(t, z(t)),$$

where $z(t) = (x(t), y(t))^t$ and $F(t, u_1, u_2) = (u_2, f(t, u_1))^t$. We managed to do so by increasing the dimension by one.

Euler's method

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Isolate $x(t+h)$:

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If t is such that $x(t)$ is known, then we can guess x at a later time $t+h$

At t_0, $x(t_0) = x_0$ is known. Thus the value of x at time $t_0 + h$ can be approximated by

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More generally, consider a partition of $[t_0, \infty)$ with points defined by

$$t_i \triangleq t_0 + ih, \quad i = 0, 1, \ldots$$

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$$x_{i+1} = x_i + hf(t_i, x_i), \quad i = 0, 1, \dots$$

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- Recursively,

$$x_i = (1 + h)^i, \quad i = 0, \ldots, n.$$

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The Global Truncation Error (GTE) we commit is:

$$\begin{aligned} GTE &\triangleq \max_{i=0,\dots,n} |x(t_i) - x_i| \\ &= \max_{i=0,\dots,n} |e^{ih} - (1+h)^i| \\ &\leq |e^{nh} - (1+h)^n| \\ &= \left| e - \left(1 + \frac{1}{n}\right)^n \right|. \end{aligned}$$

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which goes to zero as $h \to 0$ or as $n \to \infty$.

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The matrix P of the unknowns (unknown functions) is given by

$$P(s,t) = \begin{pmatrix} p_{**}(s,t) & p_{*\diamond}(s,t) & p_{*\dagger}(s,t) \\ p_{\diamond *}(s,t) & p_{\diamond\diamond}(s,t) & p_{\diamond\dagger}(s,t) \\ p_{\dagger *}(s,t) & p_{\dagger\diamond}(s,t) & p_{\dagger\dagger}(s,t) \end{pmatrix},$$

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but the last row is exactly 0 0 1 which can be omitted. Hence, we rather look at

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Example (continued)

The (matrix) vector field in this case is a linear 2×2-transformation

$$f(t, M) = M \cdot \Lambda(t),$$

where $t \geq 0$ and M is a 2×2-matrix. **NB!** Respect the order of the matrices.

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$$\Lambda(t) = \begin{pmatrix} \mu_{**}(t) & \mu_{*\diamond}(t) \\ \mu_{\diamond *}(t) & \mu_{\diamond\diamond}(t) \end{pmatrix}.$$

Hence,

$$\frac{d}{dt} P(s,t) = P(s,t)\Lambda(t), \quad t \geq 0, \quad P(s,s) = Id.$$

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Next step: discretize time and approximate $P(s, t_i)$, $i = 0, dots$.

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Euler's method gives the following scheme:

$$P_{i+1} = P_i + hP_i\Lambda(t_i) = P_i\left(Id + h\Lambda(t_i)\right) = P_i\left(Id + h\Lambda(s + ih)\right), \quad i \geq 0.$$

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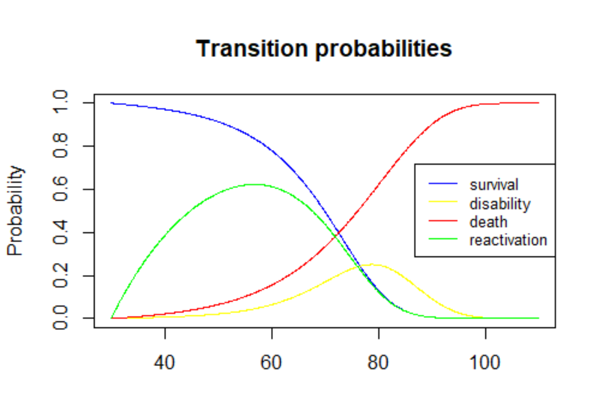
Let us use the matrix $\Lambda(t)$ from the book, see Example 2.4.2 on page 18 and Example 4.2.2 on page 30.

$$\mu_*(t) = \mu_{*\diamond}(t) + \mu_{*\dagger}(t), \quad \mu_{*\diamond}(t) = 0.0004 + 10^{0.06t-5.46},$$
$$\mu_{\diamond *}(t) = 0.05, \quad \mu_\diamond(t) = \mu_{\diamond *}(t) + \mu_{\diamond\dagger}(t).$$

Example (continued)

We start at age $s = 30$ and look at $t_i = 30 + ih$ with $h = \frac{1}{12}$ monthly steps. We run the algorithm until $t = 110$ years, i.e. $i = 0, 1, \ldots, 1080 = n$ where $n = 120 \cdot \frac{1}{h}$.



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