UiO : Department of Mathematics
University of Oslo

Life Insurance and Finance

Lecture 3: Numerical methods for ODEs (cont.)

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STK4500

Taylor’s method

Consider Taylor's approximation of x

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$$\begin{aligned} x''(t) &= \frac{d}{dt} f(t, x(t)) \\ &= \partial_t f(t, x(t)) + \partial_x f(t, x(t))[x'(t)] \\ &= \partial_t f(t, x(t)) + \partial_x f(t, x(t))[f(t, x(t))]. \end{aligned}$$

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Substituting,

$$x(t+h) = x(t) + hf(t, x(t)) + \partial_t f(t, x(t))\frac{h^2}{2} + \partial_x f(t, x(t))[f(t, x(t))]\frac{h^2}{2} + O(h^3).$$

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$\partial_M f(t, M)$ is an operator T acting on matrices such that:

$$\lim_{\substack{H \in \mathcal{M}_{2\times 2} \\ \|H\| \to 0}} \frac{\|f(t, M + H) - f(t, M) - T(H)\|}{\|H\|} = 0,$$

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where $\|\cdot\|$ denotes any matrix norm. We see that

$$\begin{aligned} \|f(t, M+H) - f(t, M) - T(H)\| &= \|(M+H)\Lambda(t) - M\Lambda(t) - T(H))\| \\ &= \|H\Lambda(t) - T(H)\| \end{aligned}$$

and hence the operator $T(H) = H\Lambda(t)$ is the derivative of $f(t, M) = M\Lambda(t)$.

Example (Disability model (cont.))

Using the matrix $P(s,t)$ in place of $x(t)$ in the formula, the Taylor expansion becomes

$$P(s,t+h) = P(s,t) + hP(s,t)\Lambda(t) + \frac{h^2}{2}P(s,t)\Lambda'(t) + \frac{h^2}{2}T(f(t,P(s,t)) + O(h^3).$$

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Further,

$$P(s,t+h) = P(s,t)\left(Id + h\Lambda(t) + \frac{h^2}{2}\Lambda'(t) + \frac{h^2}{2}\Lambda(t)^2\right) + O(h^3).$$

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One can find $\Lambda'(t)$ explicitly if possible, but here we will simple use the rough approximation $(\Lambda(t+h) - \Lambda(t))/h$. The numerical scheme finally becomes

$$P(s,t+h) \approx P(s,t)\left(Id + h\Lambda(t) + \frac{h}{2}\left[\Lambda(t+h) - \Lambda(t)\right] + \frac{h^2}{2}\Lambda(t)^2\right).$$

Example (Disability model (cont.))

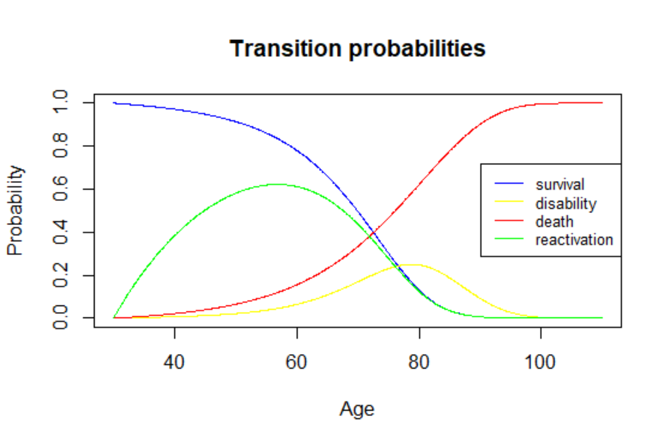

Figure: Disability model with reactivation. Second-order Taylor method: starting age $s = 30$, final age $t = 110$, step size monthly $h = 1/12$.

Example (Disability model (cont.))

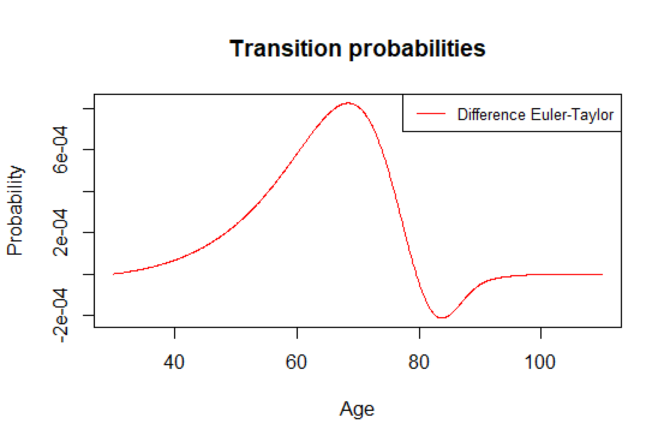

Figure: Difference $t \mapsto p_{**}^{\text{Euler}}(s,t) - p_{**}^{\text{Taylor}}(s,t)$.

Runge-Kutta method RK4

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The RK4 scheme is given by

$$x_{i+1} = x_i + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right), \quad x_0 = x(t_0),$$

where

$$\begin{aligned} k_1 &= f(t_i, x_i) \\ k_2 &= f\left(t_i + \frac{h}{2}, x_i + h\frac{k_1}{2}\right) \\ k_3 &= f\left(t_i + \frac{h}{2}, x_i + h\frac{k_2}{2}\right) \\ k_4 &= f\left(t_i + h, x_i + hk_3\right) \end{aligned}$$

Example (Disability model)

We compare all methods here below.

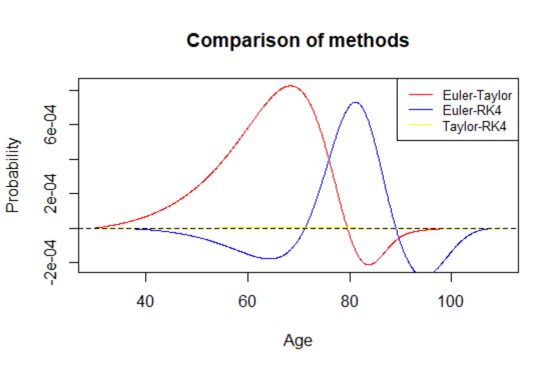

Figure: Difference between $t \mapsto p_{**}(s,t)$ for different methods.

Exercise

Implement any of the above numerical integrators for the spouse and orphan model.

Exercise

Observe that numerical methods for solving ODE's can also be applied for numerical integration. Indeed, imagine you want to find

$$x(t) = \int_{t_0}^{t} F(s)ds,$$

for some fixed t*. Then choose a grid as before* $t_i = t_0 + ih$, $i = 0, \ldots, n$ *such that* $h = \frac{t_0 - t}{n}$ *and* $f(t, x(t)) = F(t)$*. Prove that Euler's method applied to* $F(t)$ *corresponds to the left Riemann sum and Taylor's method corresponds to the trapezoidal rule. Use R or any other programming language to implement these methods and apply them to some examples.*

K2013 letter and numerical integration

Nytt dødelighetsgrunnlag i kollektiv pensjonsforsikring, usually known as *K2013* or *K13* in short, is a letter published by the Financial Supervisory Authority of Norway (Finanstilsynet) 8th of March 2013 with with the mortality basis Norwegian insurances companies and pension funds have to comply with.

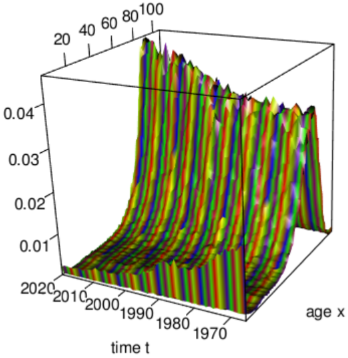

Figure: Norwegian raw mortality from 1966 to 2020 (Source: SSB).

Dødeligheten for både opplevelsesrisiko og dødsrisiko for medlemmer i forsikret bestand er gitt ved følgende formler:

$$\mu_{Kol}(x,t) = \mu_{Kol}(x,2013) * (1 + \frac{w(x)}{100})^{t-2013}$$

Her er $\mu_{Kol}(x,2013)$ dødeligheten for et medlem i alder x i 2013, mens $\mu_{Kol}(x,t)$ er dødeligheten for et medlem i alder x i kalenderår t (for t minst lik 2013). Videre benevnes dødelighetsnedgangen med $w(x)$, der

$$w(x) = \min(2{,}671548 - 0{,}172480 * x + 0{,}001485 * x^2, 0) \quad \text{for menn}$$
$$w(x) = \min(1{,}287968 - 0{,}101090 * x + 0{,}000814 * x^2, 0) \quad \text{for kvinner}$$

Dødeligheten for opplevelsesrisiko er gitt ved følgende formler:

$$1000 * \mu_{Kol}(x,2013) = (0{,}189948 + 0{,}003564 * 10^{0{,}051*x}) \quad \text{for menn}$$
$$1000 * \mu_{Kol}(x,2013) = (0{,}067109 + 0{,}002446 * 10^{0{,}051*x}) \quad \text{for kvinner}$$

Dødeligheten for dødsrisiko er gitt ved følgende formler:

$$1000 * \mu_{Kol}(x,2013) = (0{,}241752 + 0{,}004536 * 10^{0{,}051*x}) \quad \text{for menn}$$
$$1000 * \mu_{Kol}(x,2013) = (0{,}085411 + 0{,}003114 * 10^{0{,}051*x}) \quad \text{for kvinner}$$

Finanstilsynet understreker at dette er en minimumstariff og at pensjonsinnretningene selv må vurdere behovet for egne marginer utover minimumstariffen, basert på oppdatert risikostatistikk i egen forsikringsbestand.

Figure: Fragment from K2013.

Exercise (Mortality basis from Finanstilsynet)

Take $\mu(x,t)$ from K2013. Here, x is the age of the insured and t is the calendar year. As you know, mortality changes from year to year in the sense that, a person who is x today, say $t = 2023$ will not have the same mortality as a person who is x years old next year $t = 2024$.

(a) *Consider two states $\mathcal{S} = \{*, \dagger\}$ and $\mu(t) = \mu_{*\dagger}(t)$, $t \geq 0$ the mortality rate. Use Kolmogorov's equation to show that*

$$p_{**}(x+t, x+s) = \exp\left(-\int_t^s \mu(x+u)du\right), \quad s,t \geq 0, \quad s \geq t.$$

(b) *If we take $\mu(t) = \mu_{Kol}(x,t)$ where μ_{Kol} are the mortality rates from Finanstilsynet, then for a life aged x in year $Y \geq 2013$ we have*

$$p_{**}(x+t, x+s) = \exp\left(-\int_t^s \mu_{Kol}(x+u, Y+u)du\right), \quad s \geq t. \quad (2)$$

Exercise ((cont.))

In particular, the probability of surviving one more year given that one is x years old in 2023 is given by

$$p_{**}(x, x+1) = \exp\left(-\int_0^1 \mu_{Kol}(x+u, 2023+u)du\right).$$

Use Taylor's formula (of order one) to prove the (rather very rough) approximation

$$p_{**}(x, x+1) \approx \exp\left(-\mu_{Kol}(x, 2023)\right).$$

Exercise ((cont.))

(c) *In general, we prefer a more accurate integration method to find* p_{**}*. Use Riemann sums, trapezoidal rule and Simpson's method for finding an approximate value for*

$$\int_a^b f(t)dt,$$

for a Riemann integrable function f *on* $[a,b]$*.*
Apply this to $K2013$ *with mortality risk and compute*

$$p_{**}(x, x+t), \quad t \in \{0, 10, 20, 30, 40, 50\},$$

where x *is your age.*
Plot the function $t \mapsto p_{**}(x, x+t)$ *where* x *is your age.*

Exercise ((cont.))

(d) *Write an R-code which generates random lives with the mortalities given by Finanstilsynet. Plot many life times in a histogram. Compute the empirical descriptive statistics and check that they are close to the theoretical ones.*
Hint: Fix x and let T be the remaining life time of an x year old person. Then the total life time is $T_x \triangleq x + T$. What is the distribution (function) of T_x?. When you detect the distribution function of T_x, use the inverse transform sampling method to simulate values from T_x. The inverse transform method is based on the following result: if Z is a random variable with distribution function F_Z then $F_Z(Z)$ is uniformly distributed on $[0, 1]$.

Exercise (Disability income insurance)

Assume in a disability insurance that the state of the insured $X_t \in \mathcal{S}$ is described by a regular Markov chain with state space $\mathcal{S} = \{*, \diamond, \dagger\}$ where $* =$"healthy", $\diamond =$"sick" and $\dagger =$"dead". Suppose that the transition rates are given by the Gompertz-Makeham model as follows

$$\begin{aligned}
\mu_{*\diamond}(t) &= a_1 + b_1 \exp(c_1 t)\\
\mu_{\diamond *}(t) &= 0.1\mu_{*\diamond}(t),\\
\mu_{*\dagger}(t) &= a_2 + b_2 \exp(c_2 t),\\
\mu_{\diamond\dagger}(t) &= \mu_{*\dagger}(t),
\end{aligned}$$

where $a_1 = 4 \cdot 10^{-4}$, $b_1 = 3.4674 \cdot 10^{-6}$, $c_1 = 0.138155$, $a_2 = 5 \cdot 10^{-4}$, $b_2 = 7.5858 \cdot 10^{-5}$ and $c_2 = 0.087498$. Compute $p_{**}(x, x+10)$ and $p_{*\diamond}(x, x+10)$ for $x = 60$ (years). Simulate and draw the graphs of each transition probability for different values of x, say $x \in [0, 100]$.

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