UiO : Department of Mathematics
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Life Insurance and Finance

Lecture 4: Cash flows and present values

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STK4500

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Introduction

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The risk margin can for instance be

$$\mathbb{E}[L|\mathcal{F}_t] + \alpha\sqrt{\mathbb{V}[L|\mathcal{F}_t]},$$

where α is some value (loading) and $\mathbb{V}[\cdot|\mathcal{F}_t]$ stands for *conditional variance*.

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We start assuming that premiums are invested in the bank at a (deterministic) risk free rate $r(t)$ and the **only** source of uncertainty is given by X_t: the state of the insured.
Hence,

$$\mathcal{F}_t = \sigma(X_s : 0 \leq s \leq t).$$

Mathematical setting

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Define the following two processes:

1. $I_i^X(t)$ or simply $I_i(t)$ as

$$I_i^X(t) = \mathbb{I}_{\{X_t = i\}}, \quad t \geq 0, \quad i \in \mathcal{S}.$$

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- The process I_i^X tells us whether X_t is in state i or not, i.e. 1 or 0, respectively.
- The process $N_{ij}^X(t)$ counts the number of transitions from i to j on $[0, t]$.

Cash flows

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Let us imagine that we have two customers that are $x = 30$ years old and the write a pension insurance with us. Both pay premiums as long as they are active and pension will be paid out from retirement age: 70 years.

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Let us imagine that we have two customers that are $x = 30$ years old and the write a pension insurance with us. Both pay premiums as long as they are active and pension will be paid out from retirement age: 70 years.

1. We promise customer 1 a pension of 15 000NOK/month from age 70 until death.
2. We promise customer 2 a pension of 25 000NOK/month from age 70 until death.

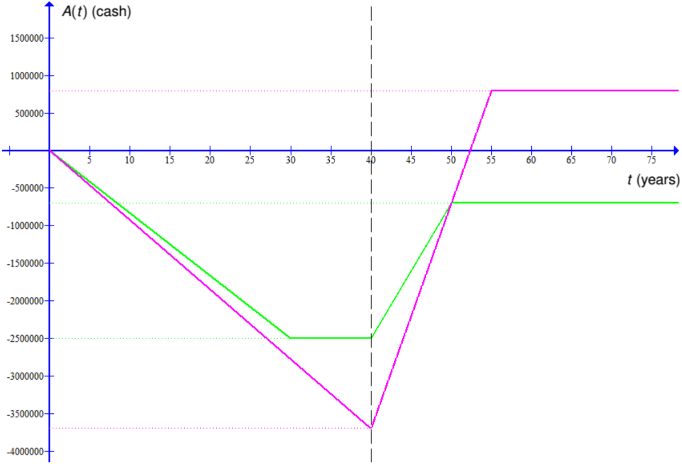

Figure: Two possible cash flows for a pension scheme of a $x = 30$ year old contributor at inception. Retirement at 70 years.

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- The pink customer passes away later than estimated, at age $30 + 55 = 85$ and we needed to finance the last 3 years which gave us a loss for this specific customer.
- **Important:** we did not deposit any money into any bank account in this example. That means, we put the money under the mattress (which is not realistic).

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Policy functions are stipulated in the policy and determine completely the type of policy. They tell us how much money to pay out to the insured at any time for each state.

Definition (Policy functions)

Let $a_i, a_{ij} : [0, \infty) \to \mathbb{R}$, $i, j \in \mathcal{S}$, $j \neq i$ be functions of bounded variation. We call them **policy functions** if they model the following quantities:

$a_i(t) =$ the accumulated payments from the insurer to the insured up to time t, given that we know that the insured has always been in state i.

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Then the (instantaneous) policy cash flow of a policy with a_i, a_{ij} is given by

$$dA(t) = \sum_j I_j^X(t) da_j(t) + \sum_{\substack{j,k \\ k \neq j}} a_{jk}(t) dN_{jk}^X(t).$$

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- The total (accumulated) cash flow is thus

$$A(t) = \sum_j \int_0^t I_j^X(s)da_j(s) + \sum_{\substack{j,k \\ k\neq j}} \int_0^t a_{jk}(s)dN_{jk}^X(s).$$

Discounting and present values of liabilities

Assume the insurer's instantaneous return on investments at time t is given by $r(t)$.

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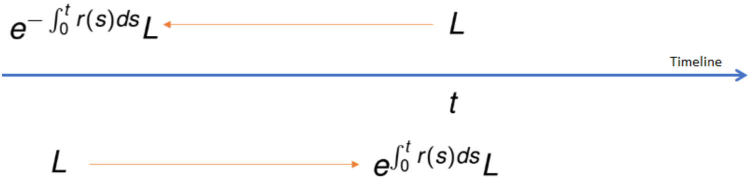

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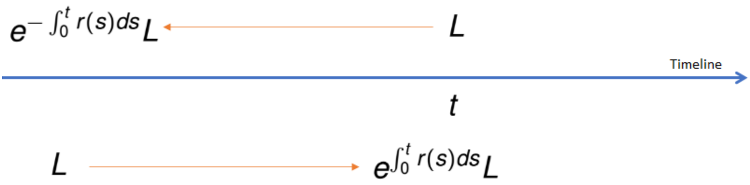

Discrete time counterpart: liability L_k at time k,

$$(1+r_k)^{-k}L_k.$$

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Summing up we get the **total cash** (liability, benefit, etc.):

$$\int_0^\infty e^{-\int_0^s r(u)du} dA(s).$$

Here, ∞ is the "end of contract", note that if the cash flow A stagnates at, say T then $dA(s) = 0$ for $s \geq T$.

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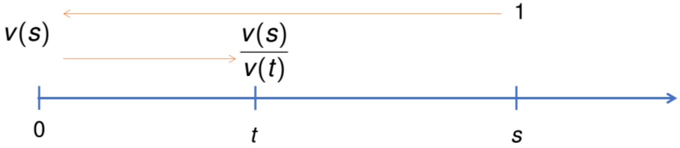
Let t be fixed. Then:



Fix t and assume that t is the new present. Then, a liability or cash flow A can be split into *past* and *future*:

$$\underbrace{\frac{1}{v(t)}\underbrace{\int_0^\infty v(s)dA(s)}_{\text{value cash flow today}}}_{\text{value cash flow at time } t} = \underbrace{\frac{1}{v(t)}\int_0^t v(s)dA(s)}_{\text{retrospective value}} + \underbrace{\frac{1}{v(t)}\int_t^\infty v(s)dA(s)}_{\text{prospective value}}.$$

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- A models cash flow of a policy
- v adjusts/discounts accordingly
- Set a present t, then we have retrospective and prospective value.

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- (Instantaneous) policy cash flow A associated to policy functions a_i and a_{ij}:

$$dA(t) = \sum_j I_j^X(t) da_j(t) + \sum_{\substack{j,k \\ k \neq j}} a_{jk}(t) dN_{jk}^X(t).$$

Summary (cont.)

- Time value corrected:

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- Assuming t is the new present. The prospective value of our liabilities is

$$V_t^+ = \frac{1}{v(t)} \sum_{j} \int_t^\infty v(s)I_j^X(s)da_j(s) + \frac{1}{v(t)} \sum_{\substack{j,k \\ k\neq j}} \int_t^\infty v(s)a_{jk}(s)dN_{jk}^X(s).$$

A quick overview on Riemann-Stieltjes integral

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You know that the Riemann integral is constructed by taking a partition of points $\{t_i\}_{i=0}^n \subset [a, b]$, $t_0 = a$, $t_n = b$ such that $\max_{i=1,\dots,n} |t_i - t_{i-1}| \to 0$ as $n \to \infty$ and

$$\int_a^b f(t)dt = \lim_n \sum_{i=1}^n f(c_i)(t_i - t_{i-1}), \quad c_i \in [t_{i-1}, t_i].$$

If $c_i = t_{i-1}$ (left/lower Riemann), if $c_i = t_i$ (right/upper Riemann).

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Then the Riemann-Stieltjes integral is just the same, replacing $t_i - t_{i-1}$ by $g(t_i) - g_(t_{i-1})$. As easy as that!

$$\int_a^b f(t)dg(t) = \lim_n \sum_{i=1}^n f(c_i)(g(t_i) - g(t_{i-1})), \quad c_i \in [t_{i-1}, t_i].$$

Assume that f is continuous then $C_f \triangleq \max_{t\in[a,b]} |f(t)| < \infty$ due to the extreme value theorem. Then

$$\left| \int_a^b f(t) dg(t) \right| \leq C_f \sup_{\mathcal{P}([a,b])} \sum_{i=1}^{n} |g(t_i) - g(t_{i-1})|.$$

Hence, a sufficient condition for $\int_a^b f(t) dg(t)$ to exist is that f is continuous and $\sup_{\mathcal{P}([a,b])} \sum_{i=1}^{n} |g(t_i) - g(t_{i-1})| < \infty$, where the supremum is taken over all possible partitions of $[a, b]$ with $\max_{i=1,\ldots,n} |t_i - t_{i-1}| \to 0$. The latter property defines a class of functions known as *functions of bounded variation*.

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Functions of bounded variation is a big class. They are functions that "cannot vary too much or too roughly". This class is far good enough for our purposes.
An example of a function of unbounded variation on any $[-a, a]$, $a > 0$ is

$$f(x) = \begin{cases} 0, \text{ if } x = 0, \\ \sin(1/x), \text{ if } x \neq 0 \end{cases},$$

while it is indeed of bounded variation on any $[a, b]$ for $0 < a < b < \infty$.

If g is a.e. differentiable with no discontinuities and g' denotes the a.e. derivative of g then it is easy to prove that (try it)

$$\int_a^b f(t)dg(t) = \int_a^b f(t)g'(t)dt,$$

But if g is a.e. differentiable with a discontinuity at say, $t = t_1 \in [a, b]$ then

$$\int_a^b f(t)dg(t) = \int_a^b f(t)g'(t)dt + f(t_1)\Delta g(t_1),$$

where $\Delta g(t) = g(t) - g(t-)$, the jump size of g at the jump time t_1.

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Example: $a_*(t) = E\mathbb{I}_{[T,\infty)}(t)$ then for any continuous function f, we have

$$\int_0^\infty f(s)da_*(s) = f(T)\Delta a_*(T) = Ef(T).$$

Examples

Example (Pure endowment insurance)

A **pure endowment** insurance is possibly one of the simplest ones. An x-year old individual enters a contract today $t = 0$ and the insurance pays them a lump sum of E monetary units if they survive up to time $T > 0$. Hence, their age goes from x to $x + T$ (it is important to distinguish between *age of the insured* and *age of the contract* whose difference is obviously x. This contract pays nothing to the insured in case they die before time T.
In some sense, this insurance is a "bet". The insured bets that they will survive to time T. The insurance company bets the opposite and pays E in case of "losing". So the relevant question is:

What is the fair price of this bet, all things taken into account?

By all things we mean: value of money (interest), mortality and amount E.

Example (Pure endowment (cont.))

How do policy functions look like in this case? Well, this insurance has only two relevant states: $*$ "alive" and $\dagger$ "dead". This insurance pays no benefits for changing from $*$ to $\dagger$, only for being in $*$, at least, passed time T. Hence, $a_{*\dagger} \equiv 0$. On the other hand, while the insured is alive during the time interval $[0, T)$ (note open interval), then the insured gets no benefits and hence, the accumulated benefits are 0. This means that $a_*(t) = 0$ for every $t \in [0, T)$. At time $t = T$ the insured gets a lump sum of E, so the accumulated benefits at time $t = T$ are obviously $a_*(T) = E$. The insured gets no further benefits passed this time, upon survival, so the accumulated payments stay at E forever. All together, this insurance is completely determined by the following policy function:

$$a_*(t) = \begin{cases} 0, & t \in [0, T) \\ E, & t \in [T, \infty) \end{cases} .$$

Example (Term insurance)

This insurance has also a specified period of time $[0, T]$ and pays a benefit B to the insured only during the time span $[0, T)$ in the only case of death. The only relevant states are $*$ and $\dagger$ and hence $a_*(t) \equiv 0$ since there are no benefits for the mere fact of being at $*$. There is one benefit for a sudden transition from $*$ to $\dagger$ if this happens during $[0, T)$. Hence, this insurance is completely determined by the following policy function:

$$a_{*\dagger}(t) = \begin{cases} B, & t \in [0, T) \\ 0, & t \in [T, \infty) \end{cases} .$$

Example (Endowment insurance)

The endowment insurance is the classic example of a life insurance. It is the sum of a pure endowment insurance and a term insurance. This means that it yields a payout in the case of an early death and also in the case of reaching the fixed age of maturity. If T is such maturity as before, then this insurance is completely determined by the following policy functions:

$$a_*(t) = \begin{cases} 0, & t \in [0, T) \\ E, & t \in [T, \infty) \end{cases} \quad , \quad a_{*\dagger}(t) = \begin{cases} B, & t \in [0, T) \\ 0, & t \in [T, \infty) \end{cases} \quad .$$

Example (Pension insurance)

This insurance has typically a period of time where the insured pays regular premiums and if a retirement age is reached, then premium payments stop and pensions are paid to the insured. If T_0 denotes the age of the contract where we start paying pensions, then if we imagine that we a pay pension P continuously (as the model setting in this section assumes) then the accumulated payments are $P \times$ "time". If $t \in [0, \infty)$ then the accumulated pension payments on $[0, T_0]$ are 0. At time $t = T_0$ we start paying the first pension P, but this happens **continuously** in time, this means that at time $t = T_0$ we have still not started, while at time $t = T_0 + \varepsilon$ a portion of εP is paid out. At time $t = T_0 + 2$ we have an accumulated payment of $2P$, and so on. Hence, the accumulated payments for being both alive and in the time span $[T_0, \infty)$ are determined by the policy function

$$a_*(t) = \begin{cases} 0, & t \in [0, T_0) \\ P(t - T_0), & t \in [T_0, \infty) \end{cases} .$$

Example (Pension insurance (cont.))

Observe that the function satisfies the discussion made above. Also, observe that in this specific case, unlike in the previous one, the function a_* is continuous and even almost everywhere differentiable with almost everywhere derivative given by

$$\dot{a}_*(t) = P.$$

Whenever our policy functions are almost everywhere differentiable with at most, a finite number of discontinuities, we will exploit this fact.

Note that this insurance model pays pensions up to infinity, since the function a_* is unbounded. In reality this is fine since nobody lives forever. But in practice, insurance companies usually set the maximum pension age to 114 or around that.

Example (Disability insurance)

This policy has three relevant states (or maybe more, but at minimum three), $*$ "alive", $\diamond$ "disabled" and $\dagger$ "dead". A disability insurance pays a periodic benefit for disability D as long as the insured is on sick leave, even from the very entry of the contract. Hence, this insurance is completely determined by the following policy function:

$$a_\diamond(t) = \begin{cases} Dt, & t \in [0, T] \\ DT, & t \in (T, \infty) \end{cases} ,$$

where T is the time where the contract expires, if any. Otherwise, T can be infinity. Note again that $a_\diamond$ models the accumulated payments during the policy time, that is why we have Dt which accumulated in time and, after T no more payments are due and hence DT, the totality of payments received, stays constant.

Exercise

Find the policy functions which completely determine a spouse insurance which pays a pension P to the remaining spouse in the case that the other passes away.

Exercise

In the examples above we have not included the payment of premiums. Let π denote a periodically paid-in premium. How would you include the payment of premiums in the policy functions? Hint: premiums are usually only paid while the insured is in state $*$ and the sign is negative according to actuarial convention.

Example of cash flow

Example: Endowment insurance

We pay out E_1 in case of survival or E_2 in case of death before T. State $*$ denotes alive and $\dagger$ deceased. Then using the formula for $A(t)$ from slide p.13,

$$A(t) = \sum_j \int_0^t I_j^X(s) da_j(s) + \sum_{\substack{j,k \\ k \neq j}} \int_0^t a_{jk}(s) dN_{jk}^X(s).$$

Then

$$A(t) = \int_0^t I_*^X(s) da_*(s) + \int_0^t a_{*\dagger}(s) dN_{*\dagger}^X(s).$$

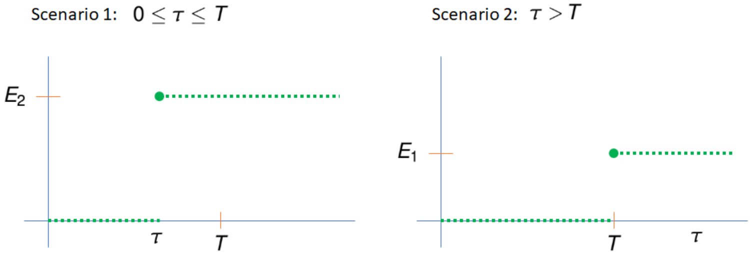
We insert the policy functions and use the property of the Riemann-Stieltjes integral from slide p.24 bottom line.

$$A(t) = E_1 I_*^X(T) \mathbb{I}_{[T,\infty)}(t) + E_2 \int_0^t \mathbb{I}_{[0,T]}(s) dN_{*\dagger}^X(s).$$

Hence,

$$A(t) = E_1 I_*^X(T) + E_2 N_{*\dagger}^X(t \wedge T).$$

If τ is the (random) death time, then



$$A(t) = \mathbb{I}_{[0,T]}(\tau)\left(\mathbb{I}_{[\tau,\infty)}(t)E_2\right) + \mathbb{I}_{(T,\infty)}(\tau)\left(\mathbb{I}_{(T,\infty)}(t)E_1\right).$$

We need to take into account **interest rate**, so we use the formula

$$L = \sum_{j} \int_0^{\infty} v(s) l_j^X(s) da_j(s) + \sum_{\substack{j,k \\ k \neq j}} \int_0^{\infty} v(s) a_{jk}(s) dN_{jk}^X(s),$$

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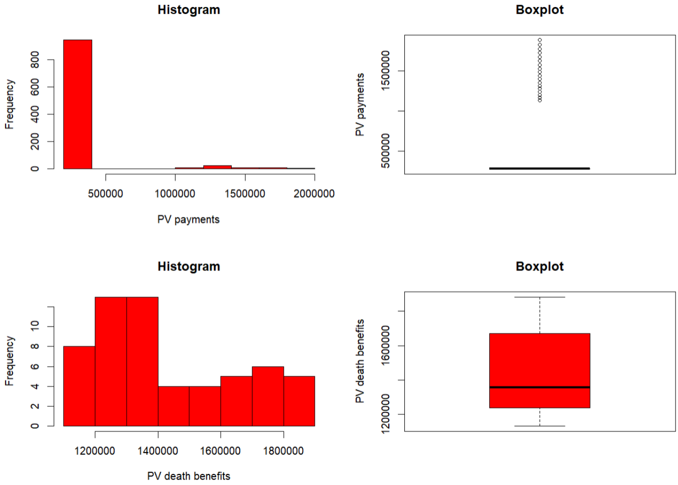
Observe that a_* jumps at T and $N_{*\dagger}$ jumps at a (random) time τ. Hence again by the property of the Riemann-Stieltjes integral,

$$L = v(T) \underbrace{I_*^X(T)}_{=\mathbb{I}_{(T,\infty)}(\tau)} \underbrace{\Delta a_*(T)}_{=E_1} + v(\tau) \underbrace{a_{*\dagger}(\tau)}_{=E_2 \mathbb{I}_{[0,T]}(\tau)} \underbrace{\Delta N_{*\dagger}^X(\tau)}_{=1}.$$

Or rather,

$$L = v(T) E_1 \mathbb{I}_{(T,\infty)}(\tau) + v(\tau) E_2 \mathbb{I}_{[0,T]}(\tau).$$

Age insured today $x_0 = 50$, $T = 20$, $E_1 = 500\,000$, $E_2 = 2\,000\,000$. Constant $r = 3\%$. Mortalities from Finanstilsynet (K2013). We obtained 5.74% of mortality in the sample (death risk). Premium: 336 651 NOK. (Exact: 337 919.8 NOK).



Final comments:

- The liability at contract inception:

$$L = v(T)E_1\mathbb{I}_{(T,\infty)}(\tau) + v(\tau)E_2\mathbb{I}_{[0,T]}(\tau)$$

is random! Because we do not know what will happen.

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- A reasonable **(net) premium** would be $\mathbb{E}[L|X_0 = *]$, i.e. expected liability given that the insured enters the contract alive.
- In the example, we computed $\mathbb{E}[L|X_0 = *]$ using a Monte-Carlo method (generated 1 000 lives), but in reality, it can be computed theoretically! (Next lecture).

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Life Insurance and Finance

Lecture 4: Cash flows and present values