UiO : Department of Mathematics
University of Oslo

Life Insurance and Finance

Lecture 5: Premiums and reserves

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STK4500

1 Review

2 Premium and reserve for the endowment policy

3 Premium and reserve (general definition)

4 Prospective reserve (explicit formula)

5 Back to endowment example

Review

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- X_t Markov chain with states i, j and transition rates μ_{ij}.

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$$I_i^X(t) : \text{tells whether } X_t \text{ is in } i \text{ or not.}$$
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- (Instantaneous) policy cash flow A associated to policy functions a_i and a_{ij}:

$$dA(t) = \sum_j I_j^X(t) da_j(t) + \sum_{\substack{j,k \\ k \neq j}} a_{jk}(t) dN_{jk}^X(t).$$

Summary (cont.)

- Time value corrected:

$$v(s)dA(s) = \sum_{j} v(s)I_j^X(s)da_j(s) + \sum_{\substack{j,k\\k\neq j}} v(s)a_{jk}(s)dN_{jk}^X(s).$$

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- Accumulated after time value correcting (present value of total future liability):

$$L = \sum_{j} \int_0^\infty v(s)I_j^X(s)da_j(s) + \sum_{\substack{j,k \\ k\neq j}} \int_0^\infty v(s)a_{jk}(s)dN_{jk}^X(s).$$

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- Assuming t is the new present. The prospective value of our liabilities is

$$V_t^+ = \frac{1}{v(t)}\sum_j \int_t^\infty v(s)I_j^X(s)da_j(s) + \frac{1}{v(t)}\sum_{\substack{j,k\\k\neq j}} \int_t^\infty v(s)a_{jk}(s)dN_{jk}^X(s).$$

Premium and reserve for the endowment policy

We start with the example: **endowment**.

Our liability at policy inception is given by

$$L = \int_0^\infty v(s) I_*^X(s) da_*(s) + \int_0^\infty v(s) a_{*\dagger}(s) dN_{*\dagger}^X(s),$$

which leads to

$$L = v(T) E_1 \mathbb{I}_{(T,\infty)}(\tau) + v(\tau) E_2 \mathbb{I}_{[0,T]}(\tau),$$

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L is a random quantity depending on the individual performance of the policy-holder.

A reasonable premium for the policy is thus

$$\pi_0 \triangleq \mathbb{E}[L | X_0 = *] \quad \textbf{(net premium)}.$$

This means, expected value of (future) liability assuming that the insured is alive at contract inception.

Let us do the computation!

What is the distribution of τ: the remaining life time of an x year old Norwegian?

$$\mathbb{P}[\tau > t] = \mathbb{P}[X_t = *|X_0 = *] = p_{**}(x, x+t) = e^{-\int_0^t \mu_{*\dagger}(x+u)du},$$

where $\mu_{*\dagger}(x+u) = \mu_{Kol}(x+u, Y+u)$. So

$$\mathbb{P}[\tau < t] = 1 - e^{-\int_0^t \mu_{*\dagger}(x+u)du}.$$

As a result, the density function of τ is given by

$$f_\tau(t) = \mu_{*\dagger}(x+t)e^{-\int_0^t \mu(x+u)du} = p_{**}(x, x+t)\mu_{*\dagger}(x+t).$$

Aim: compute

$$\pi_0 = \mathbb{E}[L|X_0 = *].$$

We have

$$L = v(T)E_1\mathbb{I}_{(T,\infty)}(\tau) + v(\tau)E_2\mathbb{I}_{[0,T]}(\tau).$$

Then

$$\begin{aligned}\mathbb{E}[L|X_0 = *] &= v(T)E_1\mathbb{E}[\mathbb{I}_{(T,\infty)}(\tau)|X_0 = *] + E_2\mathbb{E}[v(\tau)\mathbb{I}_{[0,T]}(\tau)|X_0 = *] \\ &= v(T)E_1 p_{**}(x, x+T) + E_2\mathbb{E}[v(\tau)\mathbb{I}_{[0,T]}(\tau)|X_0 = *].\end{aligned}$$

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$$\mathbb{E}[v(\tau)\mathbb{I}_{[0,T]}(\tau)|X_0 = *] = \int_0^\infty v(s)\mathbb{I}_{[0,T]}(s) f_\tau(s) ds.$$

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Altogether,

$$\boxed{\pi_0 = v(T)E_1 p_{**}(x, x+T) + E_2\int_0^T v(s)p_{**}(x, x+s)\mu_{*\dagger}(x+s)ds}.$$

The net premium of an endowment policy:

$$\pi_0 = \underbrace{v(T)E_1}_{\substack{\text{PV of } E_1 \\ \text{for } T \text{ years}}} p_{**}(x, x+T) + \int_0^T E_2 v(s) p_{**}(x, x+s) \mu_{*\dagger}(x+s) ds.$$

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Imagine that we find ourselves at time $t > 0$ and our insured is still alive. How does that change our liability?

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Imagine that we find ourselves at time $t \in (0, T]$ and our insured is still alive. How does that change our liability?

At time t the remaining liability (prospective value) is given by

$$V_t^+ = \frac{1}{v(t)} \int_t^\infty v(s) dA(s),$$

which in our case is

$$V_t^+ = \frac{v(T)}{v(t)} E_1 \mathbb{I}_{(T,\infty)}(\tau) \mathbb{I}_{[t,\infty)}(T) + \frac{v(\tau)}{v(t)} E_2 \mathbb{I}_{[t,T]}(\tau).$$

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The above quantity is what we (in this course) refer to as **prospective reserve**, that is

$$V_*^+(t) \triangleq \mathbb{E}[V_t^+ | X_t = *].$$

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We can repeat the computation for the endowment case. You will obtain:

$$V_*^+(t) = \frac{v(T)}{v(t)} E_1 p_{**}(x+t, x+T) + \int_t^T \frac{v(s)}{v(t)} E_2 p_{**}(x+t, x+s) \mu_{*\dagger}(x+s) ds.$$

Premium and reserve (general definition)

Net premium

In general, the policy's future liability **(prospective value)** is

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The one-time net premium π_0 is thus given by

$$\boxed{\pi_0 = \mathbb{E}\left[V_0^+ | X_0 = *\right]}.$$

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The *expected* remaining future liability from t, assuming that the insured is in a specific state i is given by

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Observation: The prospective reserve at time $t = 0$ and state $i = *$ is the net premium, reasonably.

Prospective reserve (explicit formula)

Take a look at the prospective value:

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Hence, we need to compute things like:

$$\mathbb{E}\left[\int_0^\infty f(s) I_j^X(s) da(s) \middle| X_t = i\right],$$

and

$$\mathbb{E}\left[\int_0^\infty f(s) dN_{jk}^X(s) \middle| X_t = i\right],$$

for functions f and a.

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for functions f and a. The first one is trivial. Let us focus on the second one.

Proof of $\mathbb{E}[\int_0^\infty f(s)dN_{jk}^X(s)|X_t = i]$.

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It is sufficient to assume that $f(s) = \mathbb{I}_{[a,b]}(s)$. Define

$$g(s) \triangleq \mathbb{E}[N_{jk}^X(s)|X_t = i], \quad s \geq t,$$

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$$\begin{aligned} g(s+h) - g(s) &= \mathbb{E}[N_{jk}^X(s+h) - N_{jk}^X(s) | X_t = i] \\ &= \sum_{l \in \mathcal{S}} \mathbb{E}[\mathbb{I}_{\{X_s = l\}} (N_{jk}^X(s+h) - N_{jk}^X(s)) | X_t = i] \end{aligned}$$

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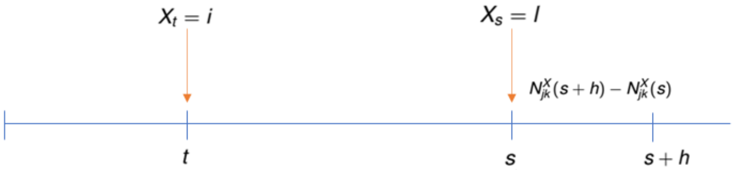
$N^X_{jk}(s+h) - N^X_{jk}(s)$ number of jumps from j to k in $[s, s+h]$ is independent of $X_t = i$, given $X_s = l$.



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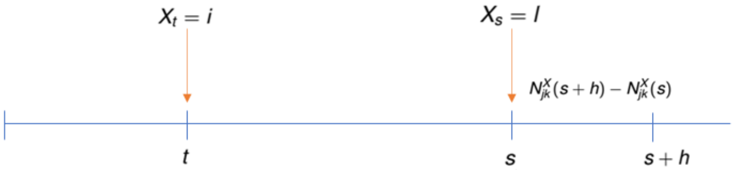

Hence,

$$
\begin{aligned}
\mathbb{E}[(N^X_{jk}(s+h) - N^X_{jk}(s))\mathbb{I}_{\{X_t=i\}}|X_s = l] = \\
&= \mathbb{E}[(N^X_{jk}(s+h) - N^X_{jk}(s))|X_s = l]\mathbb{E}[\mathbb{I}_{\{X_t=i\}}|X_s = l] \\
&= \mathbb{E}[(N^X_{jk}(s+h) - N^X_{jk}(s))|X_s = l]\mathbb{P}[X_t = i|X_s = l] \\
&= \mathbb{E}[(N^X_{jk}(s+h) - N^X_{jk}(s))|X_s = l]\frac{\mathbb{P}[X_t = i, X_s = l]}{\mathbb{P}[X_s = l]}.
\end{aligned}
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g(s+h) - g(s) &= \sum_{l \in \mathcal{S}} \frac{\mathbb{P}[X_s = l]}{\mathbb{P}[X_t = i]} \mathbb{E}[(N^X_{jk}(s+h) - N^X_{jk}(s)) \mathbb{I}_{\{X_t = i\}} | X_s = l] \\
&= \sum_{l \in \mathcal{S}} \frac{\mathbb{P}[X_s = l]}{\mathbb{P}[X_t = i]} \mathbb{E}[(N^X_{jk}(s+h) - N^X_{jk}(s)) | X_s = l] \frac{\mathbb{P}[X_t = i, X_s = l]}{\mathbb{P}[X_s = l]} \\
&= \sum_{l \in \mathcal{S}} \mathbb{E}[(N^X_{jk}(s+h) - N^X_{jk}(s)) | X_s = l] p_{il}(t,s).
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$$

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&= \sum_{l\in\mathcal{S}} \frac{\mathbb{P}[X_s=l]}{\mathbb{P}[X_t=i]}\mathbb{E}[(N_{jk}^X(s+h)-N_{jk}^X(s))|X_s=l]\frac{\mathbb{P}[X_t=i,X_s=l]}{\mathbb{P}[X_s=l]} \\
&= \sum_{l\in\mathcal{S}} \mathbb{E}[(N_{jk}^X(s+h)-N_{jk}^X(s))|X_s=l]p_{il}(t,s).
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$$

Observe that

$$
Z(h) \triangleq \mathbb{E}[(N_{jk}^X(s+h)-N_{jk}^X(s))|X_s=l] = o(h), \text{ for all } l \neq j.
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$$

Taking into account that $X_t(\omega)$ is right-continuous with left limits and $\mathcal{S}$ finite we find that

$$
\frac{Z(h)}{h} \xrightarrow{h\searrow 0} \begin{cases} \mu_{lk}(s), \text{ if } l=j, \\ 0, \text{ else} \end{cases} .
$$

Hence,

$$g'(s) = \lim_{h \to 0} \frac{g(s+h) - g(s)}{h} = p_{ij}(t,s)\mu_{jk}(s).$$

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$$g(b) - g(a) = \int_a^b g'(s)ds = \int_a^b p_{ij}(t,s)\mu_{jk}(s)ds = \int_0^\infty f(s)p_{ij}(t,s)\mu_{jk}(s)ds.$$

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On the other hand,

$$g(b) - g(a) = \mathbb{E}[N_{jk}^X(b) - N_{jk}^X(a)|X_t = i] = \mathbb{E}\left[\int_a^b dN_{jk}^X(s)|X_t = i\right]$$
$$= \mathbb{E}\left[\int_0^\infty f(s)dN_{jk}^X(s)|X_t = i\right].$$

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Therefore,

$$\mathbb{E}\left[\int_0^\infty f(s)dN_{jk}^X(s)|X_t = i\right] = \int_0^\infty f(s)p_{ij}(t,s)\mu_{jk}(s)ds,$$

for $f(s) = \mathbb{I}_{[a,b]}(s)$. Hence, for linear combinations of f as well and by a density argument, for integrable functions.

Back to the prospective value again:

$$V_t^+ = \frac{1}{v(t)} \sum_j \int_t^\infty v(s) I_j^X(s) da_j(s) + \frac{1}{v(t)} \sum_{\substack{j,k \\ k \neq j}} \int_t^\infty v(s) a_{jk}(s) dN_{jk}^X(s).$$

Apply conditional expectation $\mathbb{E}[\cdot | X_t = i]$ in order to obtain

$$\mathbb{E}[V_t^+ | X_t = i] = \frac{1}{v(t)} \sum_j \int_t^\infty v(s) \mathbb{E}[I_j^X(s) | X_t = i] da_j(s) + \frac{1}{v(t)} \sum_{\substack{j,k \\ k \neq j}} \mathbb{E}\left[\int_t^\infty v(s) a_{jk}(s) dN_{jk}^X(s) | X_t = i \right].$$

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Et voilà,

$$\boxed{V_i^+(t) = \frac{1}{v(t)} \sum_j \int_t^\infty v(s) p_{ij}(t,s) da_j(s) + \frac{1}{v(t)} \sum_{\substack{j,k \\ k \neq j}} \int_t^\infty v(s) p_{ij}(t,s) \mu_{jk}(s) a_{jk}(s) ds}.$$

Look at the formula, listen to it, what does it tell you? (Interpret).

Back to endowment example

Applying the previous theoretical formula, the **prospective reserve** for an endowment is

$$V_*^+(t) = \frac{v(T)}{v(t)} E_1 p_{**}(x+t, x+T) + E_2 \int_t^T \frac{v(s)}{v(t)} p_{**}(x+t, x+s)\mu_{*\dagger}(x+s) ds,$$

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and the **single premium**:

$$\pi_0 = v(T)E_1 p_{**}(x, x+T) + E_2 \int_0^T v(s)p_{**}(x, x+s)\mu_{*\dagger}(x+s)ds.$$

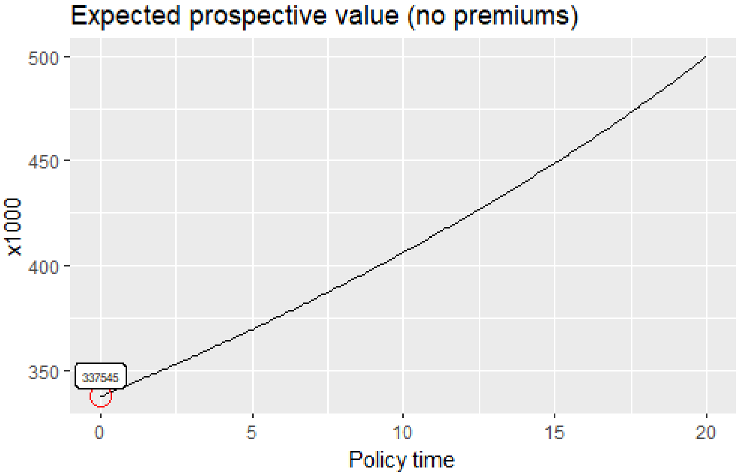

Figure: Endowment: $x = 50$, $G = 1$, $Y = 2023$, $G = 0$, $T = 20$, $r = 3\%$, $E_1 = 0.5$MNOK, $E_2 = 2$MNOK. Single premium $\pi_0 = 337\,545$NOK.

Recall the PV of the policy (endowment) is

$$V_*^+(t, A^{\text{endowment}}) = \frac{v(T)}{v(t)} E_1 p_{**}(x+t, x+T) + E_2 \int_t^T \frac{v(s)}{v(t)} p_{**}(x+t, x+s)\mu_{*\dagger}(x+s)ds.$$

It is the sum of a pure endowment and a term insurance:

$$V_*^+(t, A^{\text{pure}}) = \frac{v(T)}{v(t)} E_1 p_{**}(x+t, x+T).$$

$$V_*^+(t, A^{\text{term}}) = E_2 \int_t^T \frac{v(s)}{v(t)} p_{**}(x+t, x+s)\mu_{*\dagger}(x+s)ds.$$

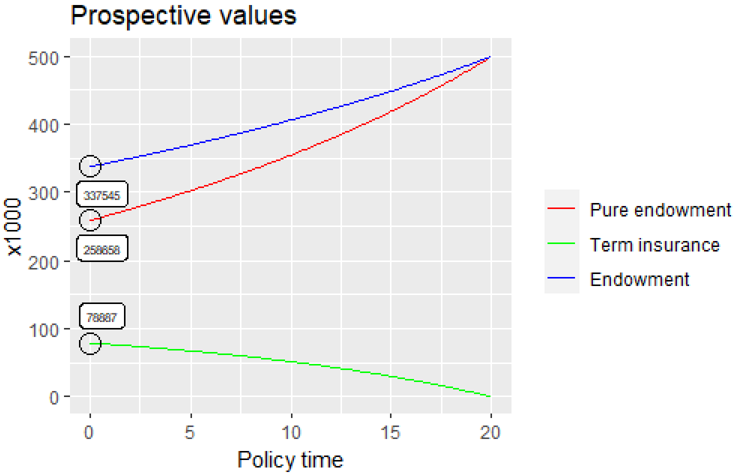

Figure: $x = 50$, $G = 1$, $Y = 2023$, $G = 0$, $T = 20$, $r = 3\%$, $E_1 = 0.5$MNOK, $E_2 = 2$MNOK. $\pi_0^{\text{term}} = 78\,887$NOK, $\pi_0^{\text{pure}} = 258\,658$NOK, $\pi_0^{\text{endowment}} = 337\,545$NOK.

Back to the endowment policy, with cash flow $A^{\text{end.}}$. Recall that the (stochastic) value of this policy is

$$V_t^+ = \frac{1}{v(t)} \int_t^T v(s) dA^{\text{end.}}(s) = \frac{v(T)}{v(t)} E_1 \mathbb{I}_{(T,\infty)}(\tau) + \frac{v(\tau)}{v(t)} E_2 \mathbb{I}_{(t,T)}(\tau).$$

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The value $V_*^+(t, A^{\text{end.}})$ is the mean of the above distribution, given the event $X_t = *$. We can also estimate the whole distribution V_t^+ for a sample of times t using Monte Carlo simulation, then look at the 95% intervals, i.e. $l_*(t), u_*(t)$ such that

$$\mathbb{P}[l_*(t) \leq V_t^+ \leq u_*(t) | X_t = *] = 0.95.$$

Then $l_*(t) \leq V_*^+(t) \leq u_*(t)$.

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David R. Banos

Life Insurance and Finance

Lecture 5: Premiums and reserves