UiO : Department of Mathematics
University of Oslo

Life Insurance and Finance

Lecture 6: Premiums, equivalence principle, prospective reserves and stress test for interest rates

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STK4500

Recap

- X_t Markov chain with states i, j and transition rates μ_{ij}.

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- (Instantaneous) policy cash flow A associated to policy functions a_i and a_{ij}:

$$dA(t) = \sum_j I_j^X(t)da_j(t) + \sum_{\substack{j,k \\ k \neq j}} a_{jk}(t)dN_{jk}^X(t).$$

- Time value corrected:

$$v(s)dA(s) = \sum_{j} v(s)I_j^X(s)da_j(s) + \sum_{\substack{j,k \\ k\neq j}} v(s)a_{jk}(s)dN_{jk}^X(s).$$

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- Accumulated after time value correcting (present value of total prospective liability):

$$L = \sum_{j} \int_0^\infty v(s)I_j^X(s)da_j(s) + \sum_{\substack{j,k \\ k\neq j}} \int_0^\infty v(s)a_{jk}(s)dN_{jk}^X(s).$$

- Assuming t is the new present.
  The **retrospective value** when seen from time t is:

$$V_t^- = \frac{1}{v(t)} \sum_j \int_0^t v(s) I_j^X(s) da_j(s) + \frac{1}{v(t)} \sum_{\substack{j,k \\ k \neq j}} \int_0^t v(s) a_{jk}(s) dN_{jk}^X(s).$$

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  The **prospective value** of our liabilities when seen from time t is

$$V_t^+ = \frac{1}{v(t)} \sum_j \int_t^\infty v(s) I_j^X(s) da_j(s) + \frac{1}{v(t)} \sum_{\substack{j,k \\ k \neq j}} \int_t^\infty v(s) a_{jk}(s) dN_{jk}^X(s).$$

Notation: we write $V_t^\pm(A)$ instead of $V_t^\pm$ to really stress the cash flow A.

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$$V_i^+(t) = \frac{1}{v(t)} \sum_j \int_t^\infty v(s) p_{ij}(t,s) da_j(s) + \frac{1}{v(t)} \sum_{\substack{j,k \\ k \neq j}} \int_t^\infty v(s) p_{ij}(t,s) \mu_{jk}(s) a_{jk}(s) ds.$$

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Notation: we write $V_i^\pm(t, A)$ instead of $V_i^\pm(t)$ to really stress the cash flow A.
Single premium: $\pi_0 = V_*^+(0)$, i.e.

$$\pi_0 = \sum_j \int_0^\infty v(s) p_{*j}(0,s) da_j(s) + \sum_{\substack{j,k \\ k \neq j}} \int_0^\infty v(s) p_{*j}(0,s) \mu_{jk}(s) a_{jk}(s) ds,$$

if $i = *$ denotes the starting state of the insured.

Premiums

Until now, we have not considered diferred payment of premiums. We know that the total expected loss at $t = 0$ when the insured is alive is $\pi_0 \triangleq V_*^+(0)$, which we reasonably call the single premium. This is the fair value of the contract that should be paid upfront.

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Paying the whole insurance upfront is sometimes inconvenient. How do we model payment through several installments? Premiums are usually paid while in the active state $*$. So they should be included in a_*.

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- Then $A = A^{\text{Prem}} + A^{\text{Ben}}$ and by linearity of the integral,

$$V_t^+(A) = V_t^+(A^{\text{Prem}}) + V_t^+(A^{\text{Ben}})$$

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and, by linearity of the expectation,

$$V_i^+(t, A) = \underbrace{V_i^+(t, A^{\text{Prem}})}_{\text{PV of premiums}} + \underbrace{V_i^+(t, A^{\text{Ben}})}_{\text{PV of benefits}} .$$

The cash flow A^{Prem} is described by the policy function

$$a_*^{\text{Prem}}(t) = \begin{cases} -\pi t, & t \in [0, T) \\ -\pi T, & t \geq T \end{cases},$$

where $*$ denotes the active state, π is the periodically (continuously) paid premium and at time T premiums are waived, T can very well be infinity.

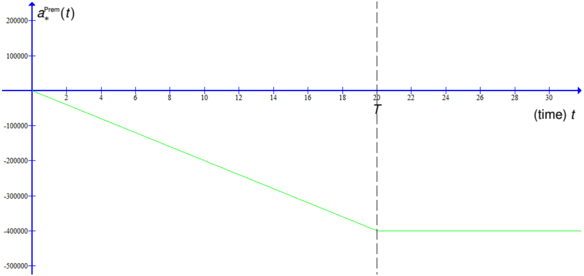

Figure: Example of policy function a_*^{Prem} where yearly premiums of π units are continuously paid for a period of $T = 20$ years and then cease.

Equivalence principle

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This means that

$$V_i^+(0, A) = V_i^+(0, A^{\text{Prem}}) + V_i^+(0, A^{\text{Ben}}),$$

for a fixed starting state $i \in \mathcal{S}$, usually $i = *$.

Example: pension

Let $x = 30$ and $G = 0$, $R = 1$ (longevity) and $Y = 2023$. Let $T_0 = 40$ years to retirement. Hence, $T_0 + x = 70$ is the retirement age. Let $T = 90$ being the maximum time of the policy. Let π be the yearly premium to be paid from age 30 to 70, i.e. for all $t \in [0, T_0)$. After T_0 we start paying yearly pensions of P until T.

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The cash flow of premiums is given by A^π with policy function

$$a_*^\pi(t) = \begin{cases} -\pi t, & t \in [0, T_0) \\ -\pi T_0, & t \geq T_0 \end{cases} .$$

The cash flow of benefits (pensions) is given by A^P with policy function

$$a_*^P(t) = \begin{cases} 0 & t \in [0, T_0) \\ P(t - T_0), & t \in [T_0, T) \\ P(T - T_0), & t \geq T \end{cases} .$$

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One can also consider both inflow and outflow of premiums and pensions, respectively, in a single policy function:

$$a_*(t) = \begin{cases} -\pi t & t \in [0, T_0) \\ -\pi T_0 + P(t - T_0), & t \in [T_0, T) \\ -\pi T_0 + P(T - T_0), & t \geq T \end{cases}.$$

Observe that this function is continuous and a.e. differentiable. The accumulated benefits after T are πT_0 premiums and $P(T - T_0)$ pensions.

The expected prospective values are given by

$$V_*^+(t,A) = V_*^+(t,A^\pi) + V_*^+(t,A^P),$$

where

$$V_*^+(t,A^\pi) = \frac{1}{v(t)} \int_t^\infty v(s) p_{**}(t,s) da_*^\pi(s)$$

and

$$V_*^+(t,A^P) = \frac{1}{v(t)} \int_t^\infty v(s) p_{**}(t,s) da_*^P(s),$$

and recall that

$$v(s) = e^{-\int_0^s r(u)du}, \quad p_{**}(t,s) = e^{-\int_t^s \mu_{Kol}(x+u,Y+u)du},$$

and μ_{Kol} is the mortality from Finanstilsynet.

Substituting the policy functions and using the rules of Riemann-Stieltjes integration we get:

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$$V_*^+(t, A^\pi) = -\pi \frac{1}{v(t)} \int_t^{T_0} v(s) p_{**}(t, s) ds \, \mathbb{I}_{[0,T_0)}(t)$$

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$$V_*^+(t, A^P) = P \frac{1}{v(t)} \int_{T_0}^{T} v(s) p_{**}(t, s) ds \, \mathbb{I}_{[0,T_0)}(t) + P \frac{1}{v(t)} \int_t^{T} v(s) p_{**}(t, s) ds \, \mathbb{I}_{[T_0,T)}(t).$$

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Altogether,

$$\begin{aligned} V_*^+(t, A) = & -\pi \frac{1}{v(t)} \int_t^{T_0} v(s) p_{**}(t, s) ds \, \mathbb{I}_{[0, T_0)}(t) \\ & + P \frac{1}{v(t)} \int_{T_0}^{T} v(s) p_{**}(t, s) ds \, \mathbb{I}_{[0, T_0)}(t) \\ & + P \frac{1}{v(t)} \int_t^{T} v(s) p_{**}(t, s) ds \, \mathbb{I}_{[T_0, T)}(t). \end{aligned}$$

Observe that, by linearity w.r.t. π, we can recast

$$V_*^+(t, A) = V_*^+(t, A^\pi) + V_*^+(t, A^P)$$

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The **equivalence principle** implies that we should choose the yearly premiums π in such a way that $V_*(0, A) = 0$.

This implies that π should be

$$\boxed{\pi = -\frac{V_*^+(0, A^P)}{V_*^+(0, A^{\pi=1})}}.$$

For the pension example, the expected prospective value at inception time is thus

$$V_*^+(0, A) = -\pi \int_0^{T_0} v(s) p_{**}(0, s) ds + P \int_{T_0}^{T} v(s) p_{**}(0, s) ds.$$

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Example: $x = 30$, $T_0 = 40$, $T = 90$, $P = 200\,000$ $r = 3\%$, $G = 0$, $R = 1$, $Y = 2023$. Then we get

$$\pi = \frac{200\,000 \int_{40}^{90} e^{-0.03s} e^{-\int_t^s \mu_{Kol}(30+u, 2023+u) du} ds}{\int_0^{40} e^{-0.03s} e^{-\int_t^s \mu_{Kol}(30+u, 2023+u) du} ds} \approx 37\,825.79 \text{NOK}.$$

Prospective reserve

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The expected prospective value of the insurance (including the prospective value of the premium inflow) is often referred to as (expected) **reserve**. It is the function $V_*^+(t, A)$ where A is the insurance cash flow including *both* premiums *and* benefits.

For the pension insurance, the (expected) reserve is given by

$$V_*^+(t, A) = V_*^+(t, A^{\text{Prem}}) + V_*^+(t, A^{\text{Ben}}),$$

i.e. the function

$$V_*^+(t, A) = -\pi \frac{1}{v(t)} \int_t^{T_0} v(s) p_{**}(t, s) ds \, \mathbb{I}_{[0,T_0)}(t)$$
$$+ P \frac{1}{v(t)} \int_{T_0}^{T} v(s) p_{**}(t, s) ds \, \mathbb{I}_{[0,T_0)}(t) + P \frac{1}{v(t)} \int_t^{T} v(s) p_{**}(t, s) ds \, \mathbb{I}_{[T_0,T)}(t).$$

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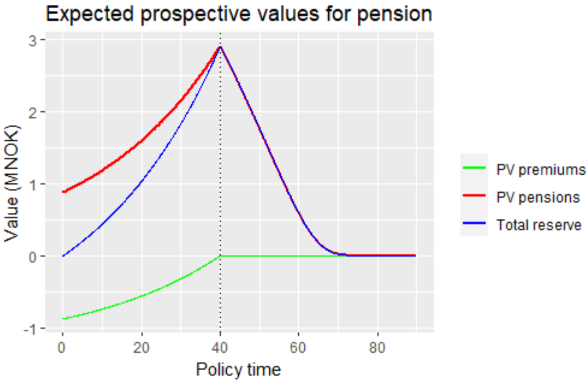

Figure: Pension $P = 0.2$MNOK, $x = 30$, $T_0 = 40$, $T = 90$, $r = 3\%$, $G = 0$, $R = 1$, $Y = 2023$, $\pi = 37\,825.79$NOK.

Example: Endowment insurance

Let A^{π}, A^{E_1} and A^{E_2} denote, respectively, the cash flows for the inflow of yearly premiums π and the outflow of survival benefit E_1 and death benefit E_2.

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Then, the expected prospective value for this insurance at time t assuming that the insured is alive is given by

$$V_*^+(t, A) = V_*^+(t, A^{\pi}) + V_*^+(t, A^{E_1}) + V_*^+(t, A^{E_2}).$$

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Let A^{π}, A^{E_1} and A^{E_2} denote, respectively, the cash flows for the inflow of yearly premiums π and the outflow of survival benefit E_1 and death benefit E_2.

From an accounting point of view, each quantity corresponds to the allocation of expected **capital** intended to cover each obligation (where premiums have a negative sign):

$$V_*^+(t,A) = \underbrace{V_*^+(t,A^{\pi})}_{\substack{\text{Allocation for}\\ \text{premiums}}} + \underbrace{V_*^+(t,A^{E_1})}_{\substack{\text{Allocation for}\\ \text{survival benefit}}} + \underbrace{V_*^+(t,A^{E_2})}_{\substack{\text{Allocation for}\\ \text{death benefit}}} .$$

Policy functions:

Premiums:

$$a_*^{\pi}(t) = \begin{cases} -\pi t, & t \in [0, T), \\ -\pi T, & t \geq T \end{cases} .$$

Survival benefit:

$$a_*^{E_1}(t) = \begin{cases} 0, & t \in [0, T), \\ E_1, & t \geq T \end{cases} .$$

Death benefit:

$$a_{*\dagger}^{E_2}(t) = \begin{cases} E_2, & t \in [0, T], \\ 0, & \text{otherwise} \end{cases} .$$

Expected prospective values:

Premiums:

$$V_*^+(t, A^\pi) = \frac{1}{v(t)} \int_t^T v(s) p_{**}(t,s) da_*^\pi(s).$$

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Death benefit:

$$V_*^+(t, A^{E_2}) = \int_t^T \frac{v(s)}{v(t)} E_2 p_{**}(t,s) \mu_{*\dagger}(s) ds.$$

Let $x = 50$ and $G = 1$, $R = 0$ and $Y = 2023$. Let $T = 20$ years to retirement. Let π be the yearly premium to be paid from age 50 to 70, i.e. for all $t \in [0, T)$. Let $E_1 = 0.5$ MNOK and $E_2 = 2$ MNOK.

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Then

$$\pi_0 = v(T)E_1 p_{**}(0, T) + \int_0^T v(s)E_2 p_{**}(0, s)\mu_{*\dagger}(s)ds = 337\,545 \text{ NOK}.$$

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The yearly premiums are obtained imposing the equivalence principle:

$$V_*^+(0, A) = \pi V_*^+(0, A^1) + V_*^+(0, A^{E_1}) + V_*^+(0, A^{E_2}) = 0.$$

Hence,

$$\pi = \frac{\pi_0}{\int_0^T v(s)p_{**}(0, s)ds} \approx 22\,846.19 \text{ NOK}.$$

Comments:

- π is paid (continuously) every year. If the actuary deposits π at $v(1) = e^r$ yearly, the investment becomes

$$\pi \sum_{k=0}^{T} e^{rk} = \pi \frac{1 - e^{rT}}{1 - e^r} = 616\,731.9 \text{ NOK}$$

at the end of the contract.

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- There is a proportion $1 - p_{**}(0, T)$ of deaths, which in this example, corresponds to: $\sim 5.74\%$.
- If we have 100 policyholders, around 94 will claim their survival benefit of 500 000 NOK and 6 will have triggered 2 000 000 each. Thus, from 616 732 we have $94 \cdot 116\,732 = 10\,972\,808$ which will be used to pay the 6 death benefits. Observe that $10\,972\,808/6 \approx 1\,828\,801$ NOK. The remaining money to reach 2 MNOK is covered with the premiums from the 6 policy holders paid.

For the endowment insurance we have,

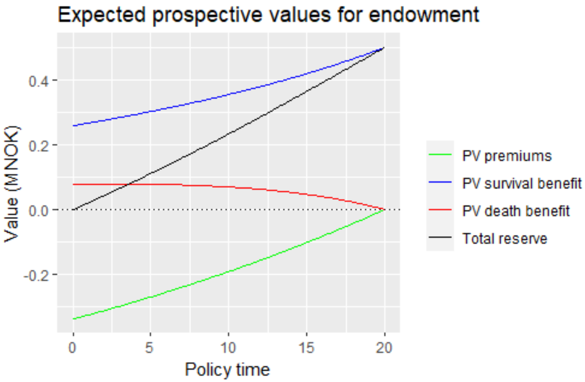

Figure: Endowment $E_1 = 0.5$ MNOK, $E_2 = 2$ MNOK, $x = 50$, $T = 20$, $r = 3\%$, $G = 1$, $R = 0$, $Y = 2023$, $\pi = 22\,846.19$ NOK.

Stress test on the (term structure of) interest rates

Wikipedia:
A stress test, in financial terminology, is an analysis or simulation designed to determine the ability of a given financial instrument or financial institution to deal with an economic crisis. Instead of doing financial projection on a "best estimate" basis, a company or its regulators may do stress testing where they look at how robust a financial instrument is in certain crashes, a form of scenario analysis.

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A **stress test** on interest rates can be found in articles 166 (increase) and 167 (decrease) in the Solvency II regulation.

Article 166

Increase in the term structure of interest rates

1. The capital requirement for the risk of an increase in the term structure of interest rates for a given currency shall be equal to the loss in the basic own funds that would result from an instantaneous increase in basic risk-free interest rates for that currency at different maturities in accordance with the following table:

Maturity (in years)	Increase
1	70 %
2	70 %
3	64 %
4	59 %
5	55 %
6	52 %
7	49 %
8	47 %

Figure: A fragment from article 166, Solvency II, supplementing Directive 2009/138/EC. See p.106 https://eur-lex.europa.eu/legal-content/EN/TXT/PDF/?uri=CELEX:32015R0035&from=EN

Article 167

Decrease in the term structure of interest rates

1. The capital requirement for the risk of a decrease in the term structure of interest rates for a given currency shall be equal to the loss in the basic own funds that would result from an instantaneous decrease in basic risk-free interest rates for that currency at different maturities in accordance with the following table:

Maturity (in years)	Decrease
1	75 %
2	65 %
3	56 %
4	50 %
5	46 %
6	42 %
7	39 %
8	36 %

Figure: A fragment from article 167, Solvency II, supplementing Directive 2009/138/EC. See p.107 https://eur-lex.europa.eu/legal-content/EN/TXT/PDF/?uri=CELEX:32015R0035&from=EN

The term structure of interest rates at a given time t is the curve of rates given by

$$T \mapsto R(t,T),$$

where $R(t,T)$ is the rate at which a bond at time t with maturity T is traded and its price is given by

$$P(t,T) = e^{-(T-t)R(t,T)}.$$

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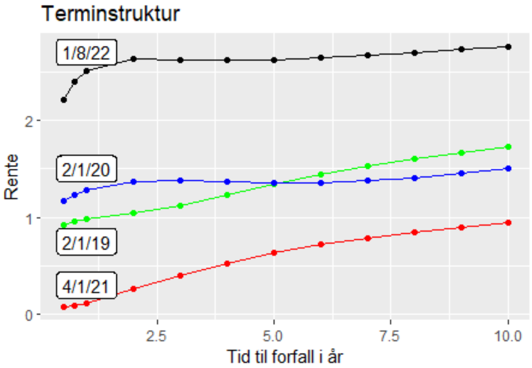

Figure: A selection of term structures for different dates t. Data from Norges Bank.

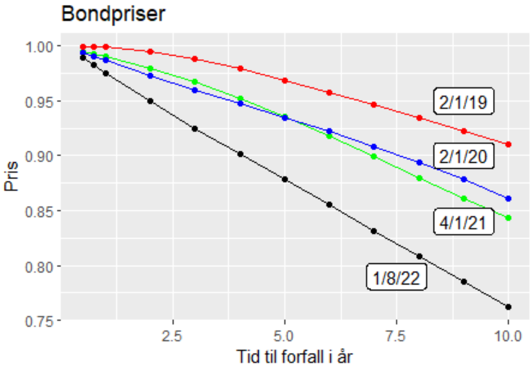

Figure: Corresponding zero-coupon bond prices for the previous term structures. Data from Norges Bank.

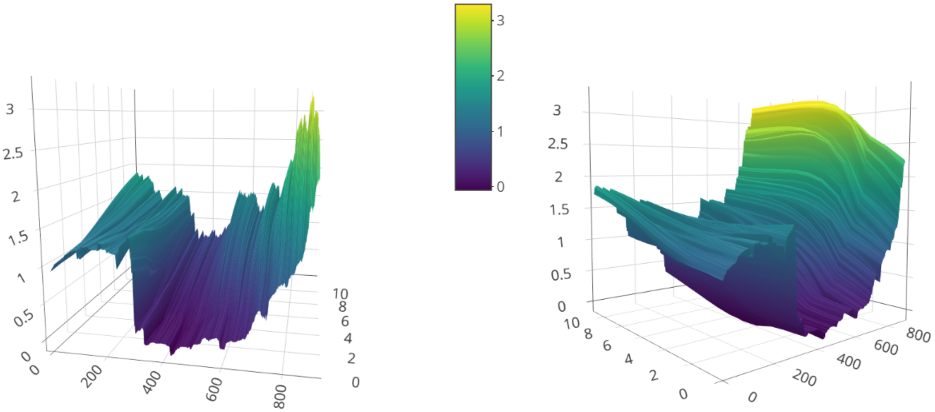

Figure: All term structures from 2/1/2019 to 1/8/2022 for maturities $0.5, 0.75, 1, 2, \ldots, 10$ years.

- You see that the rates in 2022 approach 3% above 10 years.

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- The actuary would buy X bonds with π_0 NOK and receive X NOK after $T = 20$ years. Hence, the actuary would pay

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- Solvency II tells us to stress up the 3% with maturity $T = 20$ with 26% i.e.

$$3\% + 26\% 3\% = 3.78\%$$

or decrease with

$$3\% - 29\% 3\% = 2.13\%$$

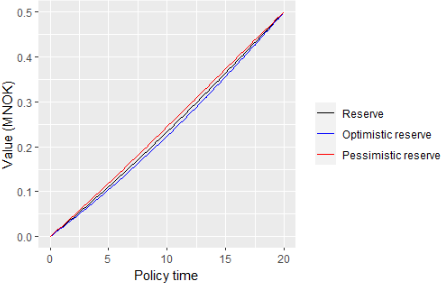

Figure: Stressed reserves

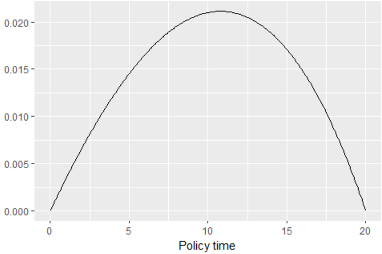

Figure: Difference

How to stress when investing in bonds continuously?

- The first premium π is used to buy $X_0 = \frac{\pi}{P(0,T)}$ bonds.

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Altogether we have bought X_0 T-bonds, X_1 $T-1$-bonds, X_2, $T-2$-bonds, ..., X_{T-1} 1-bonds, i.e.

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The rates we have bought are $R(0,T)$, $R(1,T)$, $R(2,T)$,$\dots$, $R(T-1,T)$ and the tenors are: $T, T-1, T-2, \dots, 2, 1$ years.

Recall that

$$P(t,T) = e^{-\int_t^T r(u)du} = e^{-(T-t)R(t,T)} \text{ where } R(t,T) \triangleq \frac{1}{T-t}\int_t^T r(u)du.$$

Then, stressing $R(t,T)$ means interchanging $R(t,T)$ by

$$R^*(t,T) \triangleq R(t,T) \pm w(T-t)R(t,T),$$

where $w(T-t) \in [0,1]$ and depends on the time to maturity $T-t$ of the bond. Article 166 and 167 in the Solvency II supplement give the values of w for integer $T-t$'s. The same article explains that middle values must be obtained by linear interpolation.

Observe that the stressed bond (e.g. upwards)

$$P^*(t,T) = e^{-(T-t)R^*(t,T)} = e^{-(T-t)R(t,T)}e^{-(T-t)w(T-t)R(t,T)} = \frac{v(T)}{v(t)}\left(\frac{v(T)}{v(t)}\right)^{w(T-t)}.$$

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Hence, if you assume that premiums are invested in bonds. **Stressing the interest rates** in your reserves means redoing the computations interchanging:

$$\frac{v(T)}{v(t)} \longleftrightarrow \left(\frac{v(T)}{v(t)}\right)^{1\pm w(T-t)}.$$

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University of Oslo

David R. Banos

Life Insurance and Finance

Lecture 6: Premiums, equivalence principle, prospective reserves and stress test for interest rates