UiO : Department of Mathematics
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Life Insurance and Finance

Lecture 7: Discrete time modelling

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STK4500

Introduction

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$$p_{**}(n) \triangleq p_{**}(n, n+1) = e^{-\int_n^{n+1} \mu_{Kol}(x+u, Y+u)du},$$

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- Cash flows now are accumulated amounts updated at integer times $n = 0, 1, \ldots$ and its (discrete version) prospective value is now

$$V_t^+ = \frac{1}{v(t)} \sum_{n=t}^{\infty} v(n)(A(n) - A(n-1)), \quad t \in \mathbb{N}.$$

We define the processes $I_i^X(n)$ and $N_{ij}^X(n)$ for discrete times n in an analogous way

1. $I_i^X(n)$ or simply $I_i(n)$ as

$$I_i^X(n) = \mathbb{I}_{\{X_n=i\}}, \quad n \in \mathbb{N}, \quad i \in \mathcal{S}.$$

2. $N_{ij}^X(n)$ or simply $N_{ij}(n)$ as

$$N_{ij}^X(n) = \#\{m, 0 \leq m \leq n : X_{m-1} = i, X_m = j\}, \quad n \in \mathbb{N}, \quad i, j \in \mathcal{S}.$$

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For a discrete time process Z we denote by ΔZ is increment, i.e.

$$\Delta Z(n) = Z(n+1) - Z(n), \quad n \in \mathbb{N}.$$

The increments of N_{ij}^X are 0 or 1, i.e.

$$\Delta N_{ij}^X(n) = \mathbb{I}_{\{X_{n1}=i, X_{n+1}=j\}}.$$

Definition (Policy functions in discrete time)

Let $a_i^{\text{Pre}}, a_{ij}^{\text{Post}} : \mathbb{N} \to \mathbb{R}$, $i, j \in \mathcal{S}$ be discrete time functions. We call them **policy functions** (in discrete time) if they model the following quantities:

$$a_i^{\text{Pre}}(n) = \text{payments which are due at time } n\text{, given that the insured is at time } n \text{ in } i.$$

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Notation warning! The notation Pre and Post is to indicate when the payment is made (advance vs. arrears). Remember that the function $a_i(n)$ (one index) is paid out at the beginning of the interval $[n, n+1]$ and the function $a_{ij}(n)$ (two indexes) is paid out at the end of the interval $[n, n+1]$. For this reason **we may omit** the superscripts "Pre" and "Post" when the setting is **clear**.

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&\quad \text{to } j \text{ at time } n+1.
\end{aligned}
$$

The cash produced at time n due to $a_j^{\text{Pre}}(n)$, $j \in \mathcal{S}$ is

$$\sum_j I_j^X(n) a_j^{\text{Pre}}(n).$$

The cash produced at time n due to $a_{jk}^{\text{Post}}(n)$, $j \in \mathcal{S}$ is

$$\sum_{j,k} a_{jk}^{\text{Post}}(n) \Delta N_{jk}^X(n).$$

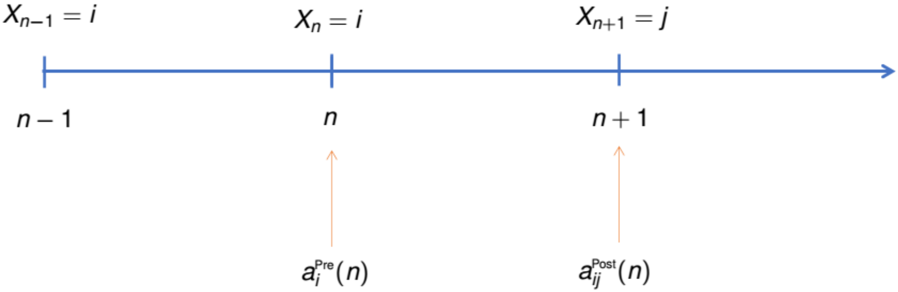
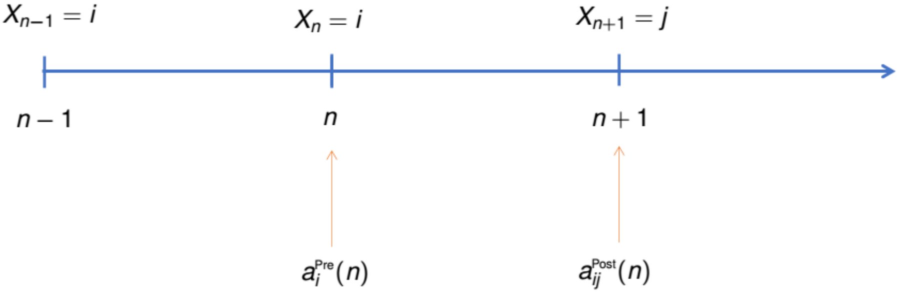


Figure: Remember that $a_i^{\text{Pre}}(n)$ is paid out at time $t = n$ for being in i while $a_{ij}^{\text{Post}}(n)$ is paid out at $t = n + 1$ when coming from i at $t = n$ and landing in j at $t = n + 1$.



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What happens when $j = i$ in $a_{ij}^{\text{Post}}(n)$? In this setting, the difference between $a_i^{\text{Pre}}(n)$ and $a_{ii}^{\text{Pre}}(n)$ is when the payment takes place. In $a_i^{\text{Pre}}(n)$ it takes place at the beginning of the time interval $[n, n+1]$ (in advance) while in $a_{ii}^{\text{Post}}(n)$ it takes place at the end of the time interval $[n, n+1]$ (in arrears).

Cash flow

1. We have that **discounted cash** from payments $a_j^{\text{Pre}}(n)$ at time n are

$$\sum_j v(n) I_j^X(n) a_j^{\text{Pre}}(n)$$

and **discounted cash** from payments $a_{jk}^{\text{Post}}(n)$ at time n are

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2. we sum up from t to end of contract to obtain the **prospective value**

$$V_t^+ = \frac{1}{v(t)} \left[\sum_{n=t}^{\infty} \sum_j v(n) I_j^X(n) a_j^{\text{Pre}}(n) + \sum_{n=t}^{\infty} \sum_{j,k} v(n+1) a_{jk}^{\text{Post}}(n) \Delta N_{jk}^X(n) \right],$$

The **expected prospective value** at discrete times $t \in \mathbb{N}$, assuming $X_t = i$, is given by

$$V_i^+(t) = \frac{1}{v(t)} \left[\sum_{n=t}^{\infty} \sum_{j} v(n) p_{ij}(t,n) a_j^{\text{Pre}}(n) + \sum_{n=t}^{\infty} \sum_{j,k} v(n+1) p_{ij}(t,n) p_{jk}(n,n+1) a_{jk}^{\text{Post}}(n) \right],$$

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Then, if we omit premiums, the expected cost of the policy (single premium π_0) in discrete time is

$$\pi_0 = V_i^+(0) = \sum_{n=0}^{\infty} \sum_j v(n) p_{ij}(0,n) a_j^{\text{Pre}}(n) + \sum_{n=0}^{\infty} \sum_{j,k} v(n+1) p_{ij}(0,n) p_{jk}(n,n+1) a_{jk}^{\text{Post}}(n).$$

Examples

Example (Pure endowment (discrete time))

Let x be the age, E the amount to be paid at a discrete time T.

Observations:

- The payment is done upon survival at discrete time $t = T$.
- The payment is made in advance, if $X_T = *$ then E is paid out immediately.
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The policy function for this insurance is

$$a_*^{\text{Pre}}(n) = \begin{cases} 0, & n \neq T, \\ E, & n = T \end{cases} .$$

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In this very particular case, one can also consider the policy function

$$a_{**}^{\text{Post}}(n) = \begin{cases} 0, & n \neq T-1, \\ E, & n = T-1 \end{cases} .$$

Example (Term insurance (discrete time))

It pays out a death benefit B if insured dies between start of contract and a time T. Observation: We have times $n = 0, 1, \ldots, T-1, T$. At $n = 0$ we assume that $X_0 = *$. If death happens during the first year then $X_0 = *$ and $X_1 = \dagger$ and payment is made at time $n = 1$. If death happens during the second year then $X_1 = *$ and $X_2 = \dagger$ and payment is made at $n = 2$, etc.

The policy function is thus

$$a_{*\dagger}^{\text{Post}}(n) = \begin{cases} B, & n = 0, 1, \ldots, T-1, \\ 0, & \text{otherwise} \end{cases} .$$

Example (Endowment insurance (discrete time))

It is a combination of the previous policies. The policy functions are given by $a_*^{\text{Pre}}(n)$ and $a_{*\dagger}^{\text{Post}}(n)$ as before.

Example (Pension insurance (discrete time))

Retirement time T_0 and end of pension time $T \geq T_0$. We pay out a pension P on the first day the insured turns $x + T_0$ years. At $n = T_0$ is $X_{T_0} = *$ then we pay out a pension P immediately. We do this every year (or month) at the beginning. Hence, the policy function is given by

$$a_*^{\text{Pre}}(n) = \begin{cases} 0, & n = 0, 1, \ldots, T_0 - 1 \\ P, & n = T_0, T_0 + 1, \ldots, T. \end{cases} \quad .$$

Example (Disability insurance)

This policy has three relevant states (or maybe more, but at minimum three), $*$ "alive", $\diamond$ "disabled" and $\dagger$ "dead". A disability insurance pays a periodic benefit for disability D as long as the insured is on sick leave, even from the very entry of the contract. Hence, this insurance is completely determined by the following policy function:

$$a_{\diamond}^{\text{Pre}}(n) = \begin{cases} D, & n = 0, 1, \ldots, T-1, \\ 0, \text{otherwise} \end{cases},$$

where T is the time where the contract expires, if any. Otherwise, T can be infinity.

Exercise

Find the policy functions which completely determine a spouse insurance which pays a pension P to the remaining spouse in the case that the other passes away, in the discrete time setting.

Exercise

In the examples above we have not included the payment of premiums. Let π denote a periodically paid-in premium. How would you include the payment of premiums in the policy functions? Hint: premiums are usually only paid while the insured is in state $*$ and the sign is negative according to actuarial convention.

Conclusion and summary:

- Discrete time Markov chain X_n with $p_{ij}(n,m)$.
- Discount factor $v(n) = e^{-rn}$ or $v(n) = (1+r)^{-n}$.
- Prospective value:

$$V_t^+ = \frac{1}{v(t)} \left[\sum_{n=t}^{\infty} \sum_j v(n) I_j^X(n) a_j^{\text{Pre}}(n) + \sum_{n=t}^{\infty} \sum_{j,k} v(n+1) a_{jk}^{\text{Post}}(n) \Delta N_{jk}^X(n) \right], \quad t \in \mathbb{N}.$$

- Expected prospective value:

$$V_i^+(t) = \frac{1}{v(t)} \left[\sum_{n=t}^{\infty} \sum_j v(n) p_{ij}(t,n) a_j^{\text{Pre}}(n) + \sum_{n=t}^{\infty} \sum_{j,k} v(n+1) p_{ij}(t,n) p_{jk}(n,n+1) a_{jk}^{\text{Post}}(n) \right],$$

  for every $t \in \mathbb{N}$.
- Like in the continuous time setting we have an equivalence principle and the possibility to include diferred payment of premiums.

Examples (in discrete time)

Insured: $x = 50$, $G = 1$, $R = 0$, $Y = 2023$. Policy: $T = 20$, $r = 3\%$, $E_1 = 0.5$ MNOK, $E_2 = 2$, MNOK.

Let A^{π} denote the cash flow of (yearly) premiums paid in advance. Let A^{E_1} denote the cash flow dealing with survival benefit and A^{E_2} death benefit.

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Premiums:

$$a_{*}^{\text{Pre},\pi}(n) = \begin{cases} -\pi, & n = 0, 1, \ldots, T-1, \\ 0, & \text{otherwise} \end{cases} .$$

Survival benefit:

$$a_{*}^{\text{Pre},E_1}(n) = \begin{cases} E_1, & n = T, \\ 0, & n \neq T \end{cases} .$$

Death benefit:

$$a_{*\dagger}^{\text{Post},E_2}(n) = \begin{cases} E_2, & n = 0, 1, \ldots, T-1, \\ 0, & \text{otherwise} \end{cases} .$$

Expected prospective values (discrete time):

Premiums:

$$V_*^+(t, A^\pi) = \frac{1}{v(t)} \sum_{n=t}^{\infty} v(n) p_{**}(t,n) a_*^{\text{Pre},\pi}(n).$$

Survival benefit:

$$V_*^+(t, A^{E_1}) = \frac{1}{v(t)} \sum_{n=t}^{\infty} v(n) p_{**}(t,n) a_*^{\text{Pre},E_1}(n).$$

Death benefit:

$$V_*^+(t, A^{E_2}) = \frac{1}{v(t)} \sum_{n=t}^{\infty} v(n+1) p_{**}(t,n) p_{*\dagger}(n,n+1) a_{*\dagger}^{\text{Post},E_2}(n).$$

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$$V_*^+(t, A^\pi) = -\pi \frac{1}{v(t)} \sum_{n=t}^{T-1} v(n) p_{**}(t,n), \quad t = 0, 1, \ldots, T-1.$$

Survival benefit:

$$V_*^+(t, A^{E_1}) = \frac{1}{v(t)} v(T) p_{**}(t,T) E_1, \quad t = 0, 1, \ldots, T.$$

Death benefit:

$$V_*^+(t, A^{E_2}) = \frac{1}{v(t)} \sum_{n=t}^{T-1} v(n+1) p_{**}(t,n) p_{*\dagger}(n, n+1) E_2, \quad t = 0, 1, \ldots, T-1.$$

Let $x = 50$ and $G = 1$, $R = 0$ and $Y = 2023$. Let $T = 20$ years to retirement. Let π be the yearly premium to be paid from age 50 to 70, i.e. for all $t \in [0, T)$. Let $E_1 = 0.5$ MNOK and $E_2 = 2$ MNOK.

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Then

$$\pi_0 = v(T)p_{**}(0, T)E_1 + \sum_{n=0}^{T-1} v(n+1)p_{**}(0, n)p_{*\dagger}(n, n+1)E_2 = 338\,752.9 \text{ NOK}.$$

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The yearly premiums are obtained imposing the equivalence principle:

$$V_*^+(0,A) = \pi V_*^+(0,A^{\pi=1}) + \underbrace{V_*^+(0,A^{E_1}) + V_*^+(0,A^{E_2})}_{=\pi_0} = 0.$$

Hence,

$$\pi = \frac{\pi_0}{\sum_{n=0}^{T-1} v(n)p_{**}(0,n)} \approx 22\,557.94 \text{ NOK}.$$

For the endowment insurance we have,

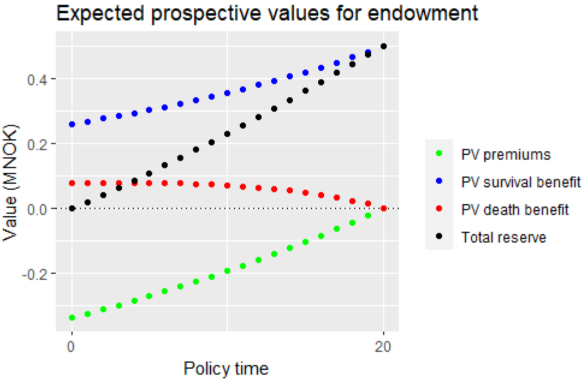

Figure: Endowment $E_1 = 0.5$ MNOK, $E_2 = 2$ MNOK, $x = 50$, $T = 20$, $r = 3\%$, $G = 1$, $R = 0$, $Y = 2023$, $\pi = 22\,557.94$ NOK.

Let us look at a **pension** scheme in **discrete time**.

Insured: $x = 30$, $G = 0$, $R = 1$, $Y = 2023$. Policy: $T_0 = 40$, $T = 90$, $r = 3\%$, $P = 200\,000$ NOK.

Let A^{π} denote the cash flow of yearly premiums paid in advance. Let A^{P} denote the cash flow dealing with the yearly pensions.

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Let A^{π} denote the cash flow of yearly premiums paid in advance. Let A^{P} denote the cash flow dealing with the yearly pensions.

Premiums:

$$a_*^{\text{Pre},\pi}(n) = \begin{cases} -\pi, & n = 0, 1, \ldots, T_0 - 1, \\ 0, & \text{otherwise} \end{cases} .$$

Pensions:

$$a_*^{\text{Pre},P}(n) = \begin{cases} P, & n = T_0, T_0 + 1, \ldots, T - 1 \\ 0, & \text{otherwise} \end{cases} .$$

Expected prospective values (discrete time):

Premiums:

$$V_*^+(t, A^\pi) = \frac{1}{v(t)} \sum_{n=t}^{\infty} v(n) p_{**}(t, n) a_*^{\text{Pre},\pi}(n).$$

Pensions:

$$V_*^+(t, A^P) = \frac{1}{v(t)} \sum_{n=t}^{\infty} v(n) p_{**}(t, n) a_*^{\text{Pre},P}(n).$$

Expected prospective values (discrete time):

Premiums:

$$V_*^+(t, A^\pi) = -\pi \frac{1}{v(t)} \sum_{n=t}^{T_0-1} v(n) p_{**}(t,n), \quad t = 0, 1, \ldots, T_0 - 1.$$

Pensions:

$$V_*^+(t, A^P) = P \frac{1}{v(t)} \sum_{n=t \vee T_0}^{T-1} v(n) p_{**}(t,n), \quad t = 0, 1, \ldots, T - 1.$$

The single premium is

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The equivalence principle established that the yearly premium should be

$$\pi = \frac{\pi_0}{\sum_{n=0}^{T_0-1} v(n) p_{**}(0, n)} = 38\,479.37 \text{ NOK}$$

For the pension we have,

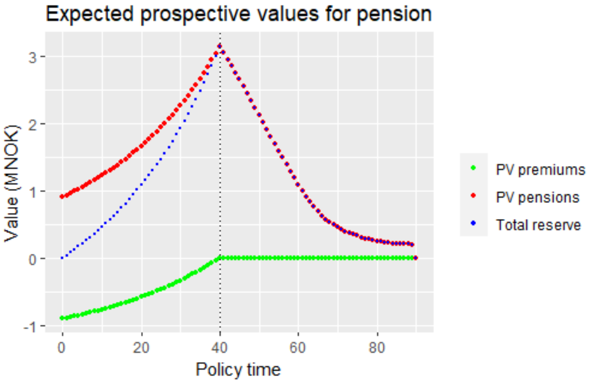

Figure: Pension $P = 0.2$ MNOK, $x = 30$, $T_0 = 40$, $T = 90$, $r = 3\%$, $G = 0$, $R = 1$, $Y = 2023$, $\pi = 38\,479.37$ NOK.

We consider the disability model from the book or the lecture notes (See slide 12 Lecture 2). Individual: $x = 30$, term $T = 40$, $r = 3\%$ and disability pension $D = 100\,000$ NOK.
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Pensions:

$$a_{\diamond}^{\text{Pre},D}(n) = \begin{cases} D, & n = 0, 1, \ldots, T-1 \\ 0, & \text{otherwise} \end{cases} .$$

Expected prospective values (discrete time):

Premiums:

$$V_*^+(t, A^\pi) = \frac{1}{v(t)} \sum_{n=t}^{\infty} v(n) p_{**}(t,n) a_*^{\text{Pre},\pi}(n).$$

Disability pensions (when healthy):

$$V_*^+(t, A^D) = \frac{1}{v(t)} \sum_{n=t}^{\infty} v(n) p_{*\diamond}(t,n) a_\diamond^{\text{Pre},D}(n).$$

Disability pensions (when disabled):

$$V_\diamond^+(t, A^D) = \frac{1}{v(t)} \sum_{n=t}^{\infty} v(n) p_{\diamond\diamond}(t,n) a_\diamond^{\text{Pre},D}(n) + \frac{1}{v(t)} \sum_{n=t}^{\infty} v(n) p_{\diamond *}(t,n) a_*^{\text{Pre},\pi}(n).$$

Expected prospective values (discrete time):

Premiums:

$$V_*^+(t, A^\pi) = -\pi \frac{1}{v(t)} \sum_{n=t}^{T-1} v(n) p_{**}(t, n).$$

Disability pensions (when healthy):

$$V_*^+(t, A^D) = D \frac{1}{v(t)} \sum_{n=t}^{T-1} v(n) p_{*\diamond}(t, n).$$

Disability pensions (when disabled):

$$V_\diamond^+(t, A^D) = D \frac{1}{v(t)} \sum_{n=t}^{T-1} v(n) p_{\diamond\diamond}(t, n) - \pi \frac{1}{v(t)} \sum_{n=t}^{T-1} v(n) p_{\diamond *}(t, n).$$

Single premium π_0 is the present value of future liabilities. When entering the insurance healthy, the single premium is:

$$\pi_0 = V_*^+(0, A) = V_*^+(0, A^D) = D \sum_{n=0}^{T-1} v(n) p_{*\diamond}(0, n) = 64\,528.99 \text{ NOK}.$$

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The actuarial equivalence principle implies

$$0 = V_*^+(0, A) = \pi V_*^+(0, A^{\pi=1}) + V_*^+(0, A^D)$$

and hence,

$$\pi = -\frac{\pi_0}{V_*^+(0, A^{\pi=1})} = \frac{64\,528.99}{\sum_{n=0}^{T-1} v(n) p_{**}(0, n)} = 3\,012.86 \text{ NOK}.$$

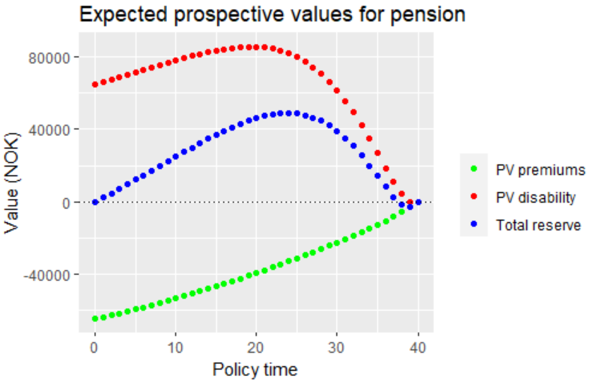

Figure: Disability $D = 0.1$ MNOK, $x = 30$, $T = 40$, $r = 3\%$, $\pi = 3\,012.86$ NOK.

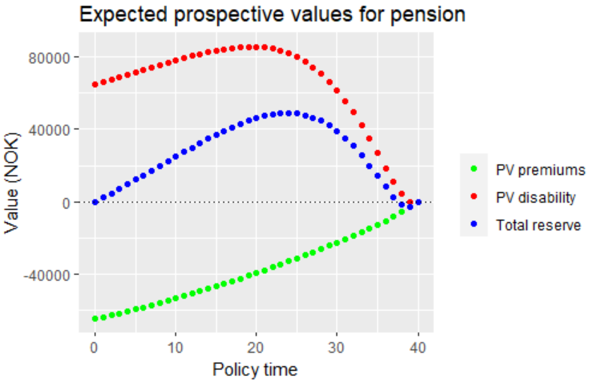

Figure: Disability $D = 0.1$ MNOK, $x = 30$, $T = 40$, $r = 3\%$, $\pi = 3\,012.86$ NOK.

The last three values of the reserve are:

$$-1\,320.14 \quad -3\,012.86 \quad 0.$$

Can you think of why and what this means?

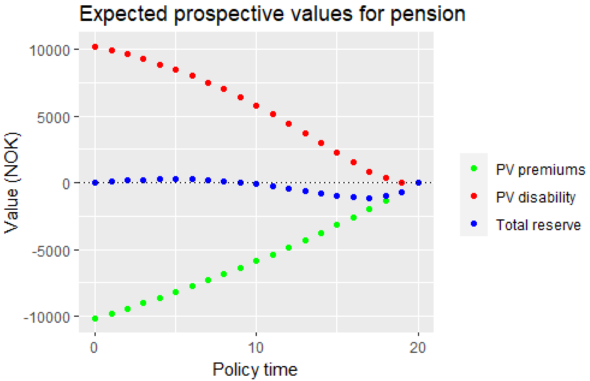

Figure: Disability $D = 0.1$ MNOK, $x = 30$, $T = 20$, $r = 3\%$, $\pi = 687.31$ NOK.

If we e.g. choose a smaller term $T = 20$ we get more negative reserves. What does that mean?

We can waive the premiums after 9 years, i.e. change

$$a_*^{\text{Pre},\pi}(n) = \begin{cases} -\pi, & n = 0, 1, \ldots, 19, \\ 0, & \text{otherwise} \end{cases}$$

by

$$a_*^{\text{Pre},\pi}(n) = \begin{cases} -\pi, & n = 0, 1, \ldots, 9, \\ 0, & \text{otherwise} \end{cases} .$$

We can waive the premiums after 10 years, i.e.

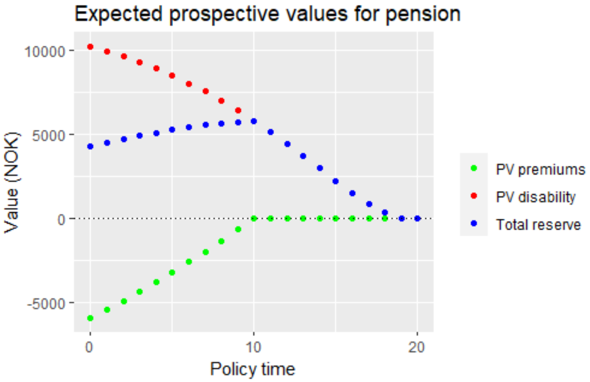

Figure: Disability $D = 0.1$ MNOK, $x = 30$, $T = 20$, $r = 3\%$, $\pi = 687.31$ NOK. Premiums are waived after ten payments.

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Life Insurance and Finance

Lecture 7: Discrete time modelling