UiO : Department of Mathematics
University of Oslo

Life Insurance and Finance

Lecture 8: Thiele's differential equation

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STK4500

Introduction

What we have learnt so far...

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- To include periodic premiums and determine their value by using the **actuarial equivalence principle**.
- All this, under a continuous time setting and a discrete time setting.
- Next:
  - Thiele's differential equation (continuous time setting)
  - Thiele's difference equation (discrete time setting)

Transition rates vs. transition probabilities

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Look at the explicit formula for *expected prospective value*:

$$V_i^+(t) = \sum_j \int_t^\infty \frac{v(s)}{v(t)} p_{ij}(t,s) da_j(s) + \sum_{\substack{j,k \\ k \neq j}} \int_t^\infty \frac{v(s)}{v(t)} p_{ij}(t,s) \mu_{jk}(s) a_{jk}(s) ds.$$

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It depends on all $p_{ij}(t,s)$ which is tricky...

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Let

$$R_i^{X^k}(h) = \int_0^h I_i^{X^k}(s)ds$$

be the time spent by individual k in state i during $[0, h]$ and

$$N_{ij}^{X^k}(h)$$

be the number of transitions $i \rightsquigarrow j$ on $[0, h]$ by individual k.

$$R_i^k(h) = \int_0^h I_i^{X^k}(s)ds \quad N_{ij}^{X^k}(h)$$

Furthermore, define the total number of time spend in i and transitions $i \rightsquigarrow j$ during $[0, h]$ as

$$R_i(h) = \sum_{k=1}^{n} R_i^{X^k}(h) \quad N_{ij}(h) = \sum_{k=1}^{n} N_{ij}^{X^k}(h).$$

Assume that μ_{ij} is constant and that we observe what happens on an interval $[0, h]$, $h > 0$.

Then the MLE estimator of μ_{ij} based on the time interval $[0, h]$ is given by

$$\boxed{\widehat{\mu}_{ij} = \widehat{\mu}_{ij}(h) = \frac{N_{ij}(h)}{R_i(h)}}$$

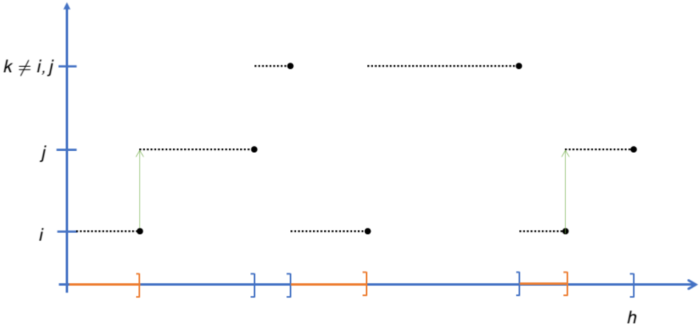

Figure: In this realization observed on $[0, h]$ we have $N_{ij} = 2$. If $h = 1$ then the orange lines account for around $R_i \approx 0.32$, so $\widehat{\mu}_{ij} = 2/0.32 = 6.25$.

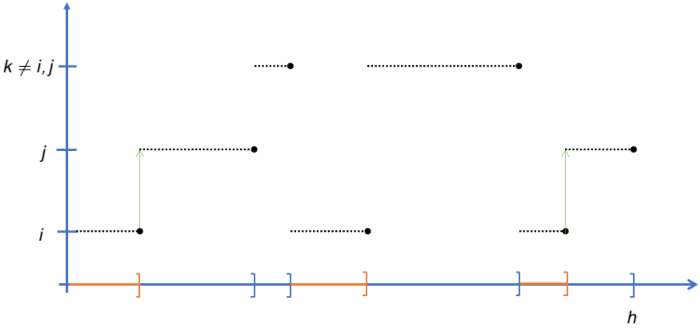

Figure: In this realization observed on $[0, h]$ we have $N_{ij} = 2$. If $h = 1$ then the orange lines account for around $R_i \approx 0.32$, so $\widehat{\mu}_{ij} = 2/0.32 = 6.25$.

In general, μ_{ij} is **not** time homogeneous. To estimate $\mu_{ij}(t)$, $t \geq 0$ one may split time into intervals where it is plausible to assume constant rates or to use a parametric family for $\mu_{ij}(t)$.

Recall that $\mu_{ij}(t)h \approx p_{ij}(t, t+h)$ for small h. Hence, estimating $\mu_{ij}(t)$ by observing what happens around t seems easier. Once we get μ_{ij} we can obtain p_{ij} through Kolmogorov's equations.

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This is the point of Thiele's equations. Let's start!

Thiele's differential equation

Recall the explicit formula for the expected prospective value, given $X_t = i$,

$$V_i^+(t) = \sum_j \int_t^\infty \frac{v(s)}{v(t)} p_{ij}(t,s) da_j(s) + \sum_{\substack{j,k \\ k \neq j}} \int_t^\infty \frac{v(s)}{v(t)} p_{ij}(t,s) \mu_{jk}(s) a_{jk}(s) ds.$$

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From now on we will assume that the policy function a_i is almost everywhere **differentiable** with at most, **one discontinuity** at maturity time T. This means $da_i(s) = \dot{a}_i(s)ds$ for a.e. s and $\Delta a_i(s) = 0$ for every $s \in [0, T)$ and $\Delta a_i(T) \neq 0$.

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Following the ingredients of Riemann-Stieltjes integration we have that for every function f:

$$\int_0^T f(s) da_i(s) = f(T) \Delta a_i(T) + \int_0^T f(s) \dot{a}_i(s) ds.$$

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Hence, under this assumption on a_i, the expected prospective value can now be written in terms of Riemann as follows:

$$V_i^+(t) = \sum_j \frac{v(T)}{v(t)} p_{ij}(t,T)\Delta a_j(T) + \sum_j \int_t^T \frac{v(s)}{v(t)} p_{ij}(t,s)\dot{a}_j(s) ds$$
$$+ \sum_{\substack{j,k \\ k\neq j}} \int_t^T \frac{v(s)}{v(t)} p_{ij}(t,s)\mu_{jk}(s) a_{jk}(s) ds.$$

Now, let us compactify things in the formula:

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Observe that the integrals in *ds* can be put under one and the sums over *j* can be merged together:

$$V_i^+(t) = \sum_j \frac{v(T)}{v(t)} p_{ij}(t,T)\Delta a_j(T)$$
$$+ \sum_j \int_t^T \frac{v(s)}{v(t)} p_{ij}(t,s) \underbrace{\left(\dot{a}_j(s)ds + \sum_{k\neq j} \mu_{jk}(s)a_{jk}(s) \right)}_{=\theta_j(s)} ds$$

So far we have

$$V_i^+(t) = \sum_j \frac{v(T)}{v(t)} p_{ij}(t,T)\Delta a_j(T) + \sum_j \int_t^T \frac{v(s)}{v(t)} p_{ij}(t,s)\theta_j(s)ds,$$

where

$$\theta_j(s) \triangleq \dot{a}_j(s) + \sum_{k\neq j} \mu_{jk}(s)a_{jk}(s).$$

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Now recall that the sum of μ_{ik} over k is 0:

$$\sum_{k} \mu_{ik}(t) = \sum_{k\neq i} \mu_{ik}(t) + \mu_{ii}(t) = 0 \iff \mu_{ii}(t) = -\sum_{k\neq i} \mu_{ik}(t).$$

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Finally, put together under the same sum:

$$\frac{d}{dt}p_{ij}(t,s) = \sum_{k\neq i} \mu_{ik}(t)\left(p_{ij}(t,s) - p_{kj}(t,s)\right).$$

Back to $V_i^+(t)$:

$$V_i^+(t) = \sum_j \frac{v(T)}{v(t)} p_{ij}(t,T)\Delta a_j(T) + \sum_j \int_t^T \frac{v(s)}{v(t)} p_{ij}(t,s)\theta_j(s)ds,$$

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Next step: pass $v(t)$ over to the left side:

$$v(t)V_i^+(t) = \sum_j v(T)p_{ij}(t,T)\Delta a_j(T) + \sum_j \int_t^T v(s)p_{ij}(t,s)\theta_j(s)ds,$$

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Next step: differentiate both sides with respect to t.

Define the following function

$$H(t) = \int_t^T f(t,s)ds$$

for an integrable function $f(t,\cdot)$ for every t. What is $H'(t)$?

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Then, clearly $H(t) = F(t, t)$. To compute $H'(t)$ we can use the (bivariate) chain rule:

$$H'(t) = \frac{\partial}{\partial x} F(x, y)|_{(x,y)=(t,t)} + \frac{\partial}{\partial y} F(x, y)|_{(x,y)=(t,t)}.$$

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We have

$$\frac{\partial}{\partial x} F(x, y) = \int_y^T \frac{\partial}{\partial x} f(x, s) ds, \quad \frac{\partial}{\partial y} F(x, y) = -f(x, y).$$

$$F(x,y) = \int_y^T f(x,s)ds \Rightarrow H(t) = F(t,t).$$

As a result,

$$H'(t) = \int_t^T \frac{\partial}{\partial x} f(x,s)|_{x=t} ds - f(t,t).$$

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In the case where $f(t,s) = v(s)p_{ij}(t,s)\theta_j(s)$ we have

$$\frac{d}{dt}\int_t^T v(s)p_{ij}(t,s)\theta_j(s)ds = \int_t^T v(s)\frac{d}{dt}p_{ij}(t,s)\theta_j(s)ds - v(t)p_{ij}(t,t)\theta_j(t).$$

Back to:

$$v(t)V_i^+(t) = \sum_j v(T)p_{ij}(t,T)\Delta a_j(T) + \sum_j \int_t^T v(s)p_{ij}(t,s)\theta_j(s)ds,$$

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The derivative of the right-hand side is:

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Recall that

$$\frac{d}{dt}p_{ij}(t,s) = \sum_{k\neq i} \mu_{ik}(t)\left(p_{ij}(t,s) - p_{kj}(t,s)\right).$$

So far,

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$$v(T)\sum_j \frac{d}{dt}p_{ij}(t,T)\Delta a_j(T) + \sum_j \int_t^T v(s)\frac{d}{dt}p_{ij}(t,s)\theta_j(s)ds - v(t)\theta_i(t).$$

Recall that

$$\frac{d}{dt}p_{ij}(t,s) = \sum_{k\neq i}\mu_{ik}(t)\left(p_{ij}(t,s) - p_{kj}(t,s)\right).$$

Let us look at the terms with $\frac{d}{dt}p_{ij}$. Substituting:

$$v(T)\sum_j \sum_{k\neq i}\mu_{ik}(t)\left(p_{ij}(t,T) - p_{kj}(t,T)\right)\Delta a_j(T)$$

$$+\sum_j \int_t^T v(s)\sum_{k\neq i}\mu_{ik}(t)\left(p_{ij}(t,s) - p_{kj}(t,s)\right)\theta_j(s)ds.$$

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$$v(T) \sum_j \sum_{k \neq i} \mu_{ik}(t) \left(p_{ij}(t,T) - p_{kj}(t,T)\right) \Delta a_j(T)$$
$$+ \sum_j \int_t^T v(s) \sum_{k \neq i} \mu_{ik}(t) \left(p_{ij}(t,s) - p_{kj}(t,s)\right) \theta_j(s) ds.$$

Indeed,

$$\sum_{k \neq i} \mu_{ik}(t) \Bigg[v(T) \sum_j \left(p_{ij}(t,T) - p_{kj}(t,T)\right) \Delta a_j(T)$$
$$+ \sum_j \int_t^T v(s) \left(p_{ij}(t,s) - p_{kj}(t,s)\right) \theta_j(s) ds \Bigg].$$

Observe that we can recover the expression for $V_i^+(t)$ and $V_k^+(t)$.

$$\sum_{k\neq i}\mu_{ik}(t)\Bigg[v(T)\sum_j\left(p_{ij}(t,T)-p_{kj}(t,T)\right)\Delta a_j(T) + \sum_j\int_t^T v(s)\left(p_{ij}(t,s)-p_{kj}(t,s)\right)\theta_j(s)ds\Bigg].$$

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So far,

$$-r(t)\cancel{v(t)}V_i^+(t)+\cancel{v(t)}\frac{d}{dt}V_i^+(t)=\cancel{v(t)}\sum_{k\neq i}\mu_{ik}(t)\left(V_i^+(t)-V_k^+(t)\right)-\cancel{v(t)}\theta_i(t).$$

In a summary,

$$-r(t)V_i^+(t) + \frac{d}{dt}V_i^+(t) = \sum_{k \neq i} \mu_{ik}(t)\left(V_i^+(t) - V_k^+(t)\right) - \theta_i(t).$$

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Remember that

$$\theta_i(t) \triangleq \dot{a}_i(t) + \sum_{k\neq i}\mu_{ik}(t)a_{ik}(t).$$

In a summary,

$$-r(t)V_i^+(t) + \frac{d}{dt}V_i^+(t) = \sum_{k\neq i} \mu_{ik}(t)\left(V_i^+(t) - V_k^+(t)\right) - \theta_i(t).$$

Remember that

$$\theta_i(t) \triangleq \dot{a}_i(t) + \sum_{k\neq i} \mu_{ik}(t) a_{ik}(t).$$

Therefore,

$$\frac{d}{dt}V_i^+(t) = r(t)V_i^+(t) - \dot{a}_i(t) + \sum_{k\neq i} \mu_{ik}(t)\left(V_i^+(t) - V_k^+(t)\right) - \sum_{k\neq i} \mu_{ik}(t) a_{ik}(t).$$

Simplifying we finally get **Thiele's differential equation**:

$$\boxed{\frac{d}{dt} V_i^+(t) = r(t) V_i^+(t) - \dot{a}_i(t) - \sum_{k \neq i} \mu_{ik}(t) \left(a_{ik}(t) + V_k^+(t) - V_i^+(t) \right)}$$

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Observe that the final condition is given by

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Remark: The equation does not depend on transition probabilities, only rates. Thus, the equation produces $V_i^+(t)$ from the rates $\{\mu_{ij}(t)\}_{ij}$ without having to go through Kolmogorov's equation. In some sense, Kolmogorov's equation is already embedded into Thiele's equation.

Examples

Pure endowment

Thiele:

$$\frac{d}{dt}V_i^+(t) = r(t)V_i^+(t) - \dot{a}_i(t) - \sum_{k \neq i} \mu_{ik}(t)\left(a_{ik}(t) + V_k^+(t) - V_i^+(t)\right).$$

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Only two states: $\mathcal{S} = \{*, \dagger\}$ and $V_\dagger^+ \equiv 0$, hence we only have one function $V_*^+(t)$. Moreover all a_{ij} are 0. The equation reduces to:

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Sine $k \neq *$ means $k = \dagger$ we have

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$$\frac{d}{dt}V_*^+(t) = r(t)V_*^+(t) - \dot{a}_*(t) - \mu_{*\dagger}(t)\left(V_\dagger^+(t) - V_*^+(t)\right).$$

Moreover, $V_\dagger \equiv 0$ and $\dot{a}_*(t) = 0$ for all $t \neq T$. Hence,

$$\frac{d}{dt}V_*^+(t) = r(t)V_*^+(t) + \mu_{*\dagger}(t)V_*^+(t), \quad V_*^+(T) = \Delta a_*(T) = E,$$

where E is the survival benefit to be paid out at time T in case of survival.

Term insurance

Thiele:

$$\frac{d}{dt} V_i^+(t) = r(t) V_i^+(t) - \dot{a}_i(t) - \sum_{k \neq i} \mu_{ik}(t) \left(a_{ik}(t) + V_k^+(t) - V_i^+(t) \right).$$

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Only two states: $\mathcal{S} = \{*, \dagger\}$ and $V_\dagger^+ \equiv 0$, hence we only have one function $V_*^+(t)$. Moreover all a_i and a_{ik} are 0 except for $a_{*\dagger}(t) = B$ for $t \in [0, T]$. The equation reduces to:

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Hence,

$$\frac{d}{dt}V_*^+(t) = r(t)V_*^+(t) + (\mu_{*\dagger}(t) + B)V_*^+(t), \quad V_*^+(T) = \Delta a_*(T) = 0,$$

Endowment insurance

Thiele:

$$\frac{d}{dt}V_i^+(t) = r(t)V_i^+(t) - \dot{a}_i(t) - \sum_{k\neq i}\mu_{ik}(t)\left(a_{ik}(t) + V_k^+(t) - V_i^+(t)\right).$$

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It's a combination of a pure endowment and a term insurance, hence:

$$\frac{d}{dt}V_*^+(t) = r(t)V_*^+(t) + (\mu_{*\dagger}(t) + B)V_*^+(t), \quad V_*^+(T) = \Delta a_*(T) = E.$$

Pension

Thiele:

$$\frac{d}{dt}V_i^+(t) = r(t)V_i^+(t) - \dot{a}_i(t) - \sum_{k\neq i}\mu_{ik}(t)\left(a_{ik}(t) + V_k^+(t) - V_i^+(t)\right).$$

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Two states:

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In a pension we have $\dot{a}_*(t) = P$ during the retirement time $t \in [T_0, T)$. Hence,

$$\frac{d}{dt}V_*^+(t) = (r(t) + \mu_{*\dagger}(t))V_*^+(t) - P\mathbb{I}_{[T_0,T)}(t), \quad V_*^+(T) = 0.$$

Premiums

Thiele:

$$\frac{d}{dt}V_i^+(t) = r(t)V_i^+(t) - \dot{a}_i(t) - \sum_{k\neq i}\mu_{ik}(t)\left(a_{ik}(t) + V_k^+(t) - V_i^+(t)\right).$$

Then we assume $\dot{a}_*(t) = -\pi\mathbb{I}_{[0,T_0)}(t)$. Hence,

$$\frac{d}{dt}V_i^+(t) = r(t)V_i^+(t) + \pi + \mu_{ik}(t)V_i^+(t), \quad V_*^+(T_0) = 0, \quad t \in [0, T_0].$$

Disability insurance

Thiele:

$$\frac{d}{dt} V_i^+(t) = r(t) V_i^+(t) - \dot{a}_i(t) - \sum_{k \neq i} \mu_{ik}(t) \left(a_{ik}(t) + V_k^+(t) - V_i^+(t) \right).$$

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Three states: $\mathcal{S} = \{*, \diamond, \dagger\}$. Hence,

$$\frac{d}{dt}V_*^+(t) = r(t)V_*^+(t) - \dot{a}_*(t) - \sum_{k\neq *}\mu_{*k}(t)\left(a_{*k}(t) + V_k^+(t) - V_*^+(t)\right),$$

$$\frac{d}{dt}V_\diamond^+(t) = r(t)V_\diamond^+(t) - \dot{a}_\diamond(t) - \sum_{k\neq \diamond}\mu_{\diamond k}(t)\left(a_{\diamond k}(t) + V_k^+(t) - V_\diamond^+(t)\right).$$

The pensions/premiums would go in $\dot{a}_*$ and the disability pensions in $\dot{a}_\diamond$.

Disability insurance

Thiele:

$$\frac{d}{dt} V_i^+(t) = r(t) V_i^+(t) - \dot{a}_i(t) - \sum_{k \neq i} \mu_{ik}(t) \left(a_{ik}(t) + V_k^+(t) - V_i^+(t)\right).$$

Three states: $\mathcal{S} = \{*, \diamond, \dagger\}$. Hence,

$$\frac{d}{dt} V_*^+(t) = r(t) V_*^+(t) - \dot{a}_*(t) - \mu_{*\diamond}(t) \left(a_{*\diamond}(t) + V_\diamond^+(t) - V_*^+(t)\right) - \mu_{*\dagger}(t) \left(a_{*\dagger}(t) + V_\dagger^+(t) - V_*^+(t)\right)$$

$$\frac{d}{dt} V_\diamond^+(t) = r(t) V_\diamond^+(t) - \dot{a}_\diamond(t) - \mu_{\diamond *}(t) \left(a_{\diamond *}(t) + V_*^+(t) - V_\diamond^+(t)\right) - \mu_{\diamond\dagger}(t) \left(a_{\diamond\dagger}(t) + V_\dagger^+(t) - V_\diamond^+(t)\right)$$

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$$\frac{d}{dt}V_\diamond^+(t) = r(t)V_\diamond^+(t) - \dot{a}_\diamond(t) - \mu_{\diamond *}(t)\left(a_{\diamond *}(t) + V_*^+(t) - V_\diamond^+(t)\right) - \mu_{\diamond\dagger}(t)\left(a_{\diamond\dagger}(t) - V_\diamond^+(t)\right).$$

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David R. Banos

Life Insurance and Finance

Lecture 8: Thiele's differential equation