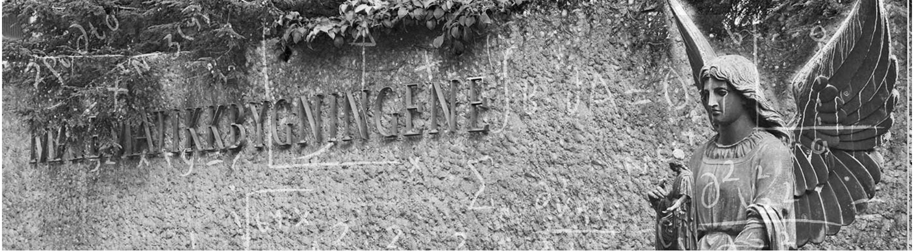

UiO : Department of Mathematics
University of Oslo

Life Insurance and Finance

Lecture 9: Thiele's difference equation

David R. Banos

STK4500

Introduction

What we have learnt so far...

- To model the **states** of the insured with a Markov chain X_t.

What we have learnt so far...

- To model the **states** of the insured with a Markov chain X_t.
- To model **cash flows**, denoted by A, with **policy functions** according to the states of the insured.

What we have learnt so far...

- To model the **states** of the insured with a Markov chain X_t.
- To model **cash flows**, denoted by A, with **policy functions** according to the states of the insured.
- To defined **stochastic prospective value** V_t^+ and the **expected prospective value** given a state i, $V_i^+(t)$.

What we have learnt so far...

- To model the **states** of the insured with a Markov chain X_t.
- To model **cash flows**, denoted by A, with **policy functions** according to the states of the insured.
- To defined **stochastic prospective value** V_t^+ and the **expected prospective value** given a state i, $V_i^+(t)$.
- To include periodic premiums and determine their value by using the **actuarial equivalence principle**.

What we have learnt so far...

- To model the **states** of the insured with a Markov chain X_t.
- To model **cash flows**, denoted by A, with **policy functions** according to the states of the insured.
- To defined **stochastic prospective value** V_t^+ and the **expected prospective value** given a state i, $V_i^+(t)$.
- To include periodic premiums and determine their value by using the **actuarial equivalence principle**.
- All this, under a continuous time setting and a discrete time setting.

What we have learnt so far...

- To model the **states** of the insured with a Markov chain X_t.
- To model **cash flows**, denoted by A, with **policy functions** according to the states of the insured.
- To defined **stochastic prospective value** V_t^+ and the **expected prospective value** given a state i, $V_i^+(t)$.
- To include periodic premiums and determine their value by using the **actuarial equivalence principle**.
- All this, under a continuous time setting and a discrete time setting.
- Thiele's differential equation (continuous time setting) for $V_i^+(t)$, $t \geq 0$.

What we have learnt so far...

- To model the **states** of the insured with a Markov chain X_t.
- To model **cash flows**, denoted by A, with **policy functions** according to the states of the insured.
- To defined **stochastic prospective value** V_t^+ and the **expected prospective value** given a state i, $V_i^+(t)$.
- To include periodic premiums and determine their value by using the **actuarial equivalence principle**.
- All this, under a continuous time setting and a discrete time setting.
- Thiele's differential equation (continuous time setting) for $V_i^+(t)$, $t \geq 0$.
- Next:
  - Thiele's difference equation (discrete time setting) for $V_i^+(t)$, $t \in \mathbb{N}$.
  - Examples.

Thiele’s difference equation

Recall the **expected prospective value** at discrete times $t \in \mathbb{N}$, assuming $X_t = i$, is given by

$$V_i^+(t) = \frac{1}{v(t)} \left[\sum_j \sum_{n=t}^{\infty} v(n) p_{ij}(t,n) a_j^{\text{Pre}}(n) + \sum_{j,k} \sum_{n=t}^{\infty} v(n+1) p_{ij}(t,n) p_{jk}(n,n+1) a_{jk}^{\text{Post}}(n) \right],$$

Recall the **expected prospective value** at discrete times $t \in \mathbb{N}$, assuming $X_t = i$, is given by

$$V_i^+(t) = \frac{1}{v(t)} \left[\sum_j \sum_{n=t}^{\infty} v(n) p_{ij}(t,n) a_j^{\text{Pre}}(n) + \sum_{j,k} \sum_{n=t}^{\infty} v(n+1) p_{ij}(t,n) p_{jk}(n,n+1) a_{jk}^{\text{Post}}(n) \right],$$

Steps to prove Thiele's difference equation:

1. Compactify the expression for $V_i^+(t)$, $t \in \mathbb{N}$.
2. Separate the sum over n into *one* summand $n = t$ and the rest $n \geq t+1$.
3. Use Chapman-Kolmogorov equation to go from t to n via $t+1$, i.e.

$$p_{ij}(t,n) = \sum_k p_{ik}(t,t+1) p_{kj}(t+1,n).$$

4. Put things back together.

Recall the **expected prospective value** at discrete times $t \in \mathbb{N}$, assuming $X_t = i$, is given by

$$V_i^+(t) = \frac{1}{v(t)} \left[\sum_j \sum_{n=t}^{\infty} v(n) p_{ij}(t,n) a_j^{\text{Pre}}(n) + \sum_{j,k} \sum_{n=t}^{\infty} v(n+1) p_{ij}(t,n) p_{jk}(n,n+1) a_{jk}^{\text{Post}}(n) \right],$$

We can express $V_i^+(t)$ as

$$V_i^+(t) = \frac{1}{v(t)} \sum_j \sum_{n=t}^{\infty} v(n) p_{ij}(t,n) \underbrace{\left(a_j^{\text{Pre}}(n) + \sum_k \frac{v(n+1)}{v(n)} p_{jk}(n,n+1) a_{jk}^{\text{Post}}(n) \right)}_{=\theta_j(n)}.$$

Recall the **expected prospective value** at discrete times $t \in \mathbb{N}$, assuming $X_t = i$, is given by

$$V_i^+(t) = \frac{1}{v(t)} \left[\sum_j \sum_{n=t}^{\infty} v(n) p_{ij}(t,n) a_j^{\text{Pre}}(n) + \sum_{j,k} \sum_{n=t}^{\infty} v(n+1) p_{ij}(t,n) p_{jk}(n,n+1) a_{jk}^{\text{Post}}(n) \right],$$

We can express $V_i^+(t)$ as

$$V_i^+(t) = \frac{1}{v(t)} \sum_j \sum_{n=t}^{\infty} v(n) p_{ij}(t,n) \underbrace{\left(a_j^{\text{Pre}}(n) + \sum_k \frac{v(n+1)}{v(n)} p_{jk}(n,n+1) a_{jk}^{\text{Post}}(n) \right)}_{=\theta_j(n)}.$$

That is,

$$V_i^+(t) = \frac{1}{v(t)} \sum_j \sum_{n=t}^{\infty} v(n) p_{ij}(t,n) \theta_j(n),$$

where

$$\theta_j(n) \triangleq a_j^{\text{Pre}}(n) + \sum_k v_n p_{jk}(n,n+1) a_{jk}^{\text{Post}}(n),$$

where $v_n = \frac{v(n+1)}{v(n)}$ is the one-step discount factor.

$$V_i^+(t) = \frac{1}{v(t)} \sum_j \sum_{n=t}^{\infty} v(n) p_{ij}(t,n) \theta_j(n),$$

where

$$\theta_j(n) \triangleq a_j^{\text{Pre}}(n) + \sum_k v_n p_{jk}(n, n+1) a_{jk}^{\text{Post}}(n).$$

Now, separate the sum $\sum_{n=t}^{\infty} a(n) = a(t) + \sum_{n=t+1}^{\infty} a(n)$.

$$\begin{aligned} V_i^+(t) &= \frac{1}{v(t)} \left[\sum_j v(t) \underbrace{p_{ij}(t,t)}_{\mathbf{1}_{j=i}} \theta_j(t) + \sum_j \sum_{n=t+1}^{\infty} v(n) p_{ij}(t,n) \theta_j(n) \right] \\ &= \frac{1}{v(t)} \left[v(t) \theta_i(t) + \sum_j \sum_{n=t+1}^{\infty} v(n) p_{ij}(t,n) \theta_j(n) \right] \\ &= \theta_i(t) + \frac{1}{v(t)} \sum_j \sum_{n=t+1}^{\infty} v(n) p_{ij}(t,n) \theta_j(n). \end{aligned}$$

So far,

$$V_i^+(t) = \theta_i(t) + \frac{1}{v(t)} \sum_j \sum_{n=t+1}^{\infty} v(n) p_{ij}(t,n) \theta_j(n),$$

where

$$\theta_j(n) \triangleq a_j^{\text{Pre}}(n) + \sum_k v_n p_{jk}(n, n+1) a_{jk}^{\text{Post}}(n).$$

So far,

$$V_i^+(t) = \theta_i(t) + \frac{1}{v(t)} \sum_j \sum_{n=t+1}^{\infty} v(n) p_{ij}(t,n) \theta_j(n),$$

where

$$\theta_j(n) \triangleq a_j^{\text{Pre}}(n) + \sum_k v_n p_{jk}(n, n+1) a_{jk}^{\text{Post}}(n).$$

Now use Chapman-Kolmogorov equation to go from t to n via $t+1$, that is $p_{ij}(t,n) = \sum_k p_{ik}(t, t+1) p_{kj}(t+1, n)$. Then,

$$V_i^+(t) = \theta_i(t) + \frac{1}{v(t)} \sum_j \sum_{n=t+1}^{\infty} v(n) \sum_k p_{ik}(t, t+1) p_{kj}(t+1, n) \theta_j(n).$$

So far,

$$V_i^+(t) = \theta_i(t) + \frac{1}{v(t)} \sum_j \sum_{n=t+1}^{\infty} v(n) p_{ij}(t,n) \theta_j(n),$$

where

$$\theta_j(n) \triangleq a_j^{\text{Pre}}(n) + \sum_k v_n p_{jk}(n, n+1) a_{jk}^{\text{Post}}(n).$$

Now use Chapman-Kolmogorov equation to go from t to n via $t+1$, that is $p_{ij}(t,n) = \sum_k p_{ik}(t, t+1) p_{kj}(t+1, n)$. Then,

$$V_i^+(t) = \theta_i(t) + \frac{1}{v(t)} \sum_j \sum_{n=t+1}^{\infty} v(n) \sum_k p_{ik}(t, t+1) p_{kj}(t+1, n) \theta_j(n).$$

Take the sum over k as far out as you can and everything that depends upon k:

$$V_i^+(t) = \theta_i(t) + \sum_k p_{ik}(t, t+1) \frac{1}{v(t)} \underbrace{\sum_j \sum_{n=t+1}^{\infty} v(n) p_{kj}(t+1, n) \theta_j(n)}_{= v(t+1) V_k^+(t+1)}.$$

Thus,

$$V_i^+(t) = \theta_i(t) + \sum_k p_{ik}(t, t+1) \underbrace{\frac{v(t+1)}{v(t)}}_{=v_t} V_k^+(t+1),$$

where v_t is the one-step discount factor and

$$\theta_j(t) \triangleq a_j^{\text{Pre}}(t) + \sum_k v_t p_{jk}(t, t+1) a_{jk}^{\text{Post}}(t).$$

Thus,

$$V_i^+(t) = \theta_i(t) + \sum_k p_{ik}(t, t+1) \underbrace{\frac{v(t+1)}{v(t)}}_{=v_t} V_k^+(t+1),$$

where v_t is the one-step discount factor and

$$\theta_j(t) \triangleq a_j^{\text{Pre}}(t) + \sum_k v_t p_{jk}(t, t+1) a_{jk}^{\text{Post}}(t).$$

Putting things together and taking common factor we obtain the recursion formula for every $t \in \mathbb{N}$:

$$\boxed{V_i^+(t) = a_i^{\text{Pre}}(t) + \sum_k v_t p_{ik}(t, t+1) \left(a_{ik}^{\text{Post}}(t) + V_k^+(t+1)\right)}$$

Examples

Premiums (only)

Thiele:

$$V_i^+(t) = a_i^{\text{Pre}}(t) + \sum_j v_t p_{ij}(t, t+1) \left(a_{ij}^{\text{Post}}(t) + V_j^+(t+1) \right)$$

Premiums (only)

Thiele:

$$V_i^+(t) = a_i^{\text{Pre}}(t) + \sum_j v_t p_{ij}(t, t+1)\left(a_{ij}^{\text{Post}}(t) + V_j^+(t+1)\right)$$

Only two states: $\mathcal{S} = \{*, \dagger\}$ and $V_\dagger^+ \equiv 0$, hence we only have one function $V_*^+(t)$. Moreover all a_{ij}^{Post} are 0. Premiums of -1 are allocated in the function $a_*^{\text{Pre}} = -1$ The equation reduces to:

Premiums (only)

Thiele:

$$V_i^+(t) = a_i^{\text{Pre}}(t) + \sum_j v_t p_{ij}(t, t+1)\left(a_{ij}^{\text{Post}}(t) + V_j^+(t+1)\right)$$

Only two states: $\mathcal{S} = \{*, \dagger\}$ and $V_\dagger^+ \equiv 0$, hence we only have one function $V_*^+(t)$. Moreover all a_{ij}^{Post} are 0. Premiums of -1 are allocated in the function $a_*^{\text{Pre}} = -1$ The equation reduces to:

$$V_*^+(t) = a_*^{\text{Pre}}(t) + v_t p_{**}(t, t+1) V_*^+(t+1), \quad t \in \{0, 1, 2, \ldots, T_0 - 1\}.$$

Premiums (only)

Thiele:

$$V_i^+(t) = a_i^{\text{Pre}}(t) + \sum_j v_t p_{ij}(t, t+1)\left(a_{ij}^{\text{Post}}(t) + V_j^+(t+1)\right)$$

Only two states: $\mathcal{S} = \{*, \dagger\}$ and $V_\dagger^+ \equiv 0$, hence we only have one function $V_*^+(t)$. Moreover all a_{ij}^{Post} are 0. Premiums of -1 are allocated in the function $a_*^{\text{Pre}} = -1$ The equation reduces to:

$$V_*^+(t) = a_*^{\text{Pre}}(t) + v_t p_{**}(t, t+1) V_*^+(t+1), \quad t \in \{0, 1, 2, \ldots, T_0 - 1\}.$$

Moreover, the final condition is given as $V_*^+(T_0) = 0$ where $T_0 - 1$ is the last time the premium is paid in.

Premiums (only)

Thiele:

$$V_i^+(t) = a_i^{\text{Pre}}(t) + \sum_j v_t p_{ij}(t, t+1)\left(a_{ij}^{\text{Post}}(t) + V_j^+(t+1)\right)$$

Only two states: $\mathcal{S} = \{*, \dagger\}$ and $V_\dagger^+ \equiv 0$, hence we only have one function $V_*^+(t)$. Moreover all a_{ij}^{Post} are 0. Premiums of -1 are allocated in the function $a_*^{\text{Pre}} = -1$ The equation reduces to:

$$V_*^+(t) = a_*^{\text{Pre}}(t) + v_t p_{**}(t, t+1) V_*^+(t+1), \quad t \in \{0, 1, 2, \dots, T_0 - 1\}.$$

Moreover, the final condition is given as $V_*^+(T_0) = 0$ where $T_0 - 1$ is the last time the premium is paid in.

Procedure:

1. From $t = T_0$ you have $V_*^+(T_0) = 0$.
2. Then go one step back to $t = T_0 - 1$, to compute $V_*^+(T_0 - 1)$ recursively:
$$V_*^+(T_0 - 1) = \underbrace{a_*^{\text{Pre}}(T_0 - 1)}_{-1} + v_{T_0-1} p_{**}(T_0 - 1, T_0) V_*^+(T_0).$$
3. Now you have $V_*^+(T_0 - 1)$ you can go on to computing $V_*^+(T_0 - 2)$ and so on down to $V_*^+(0)$.

Pure endowment

Thiele:

$$V_i^+(t) = a_i^{\text{Pre}}(t) + \sum_j v_t p_{ij}(t, t+1)\left(a_{ij}^{\text{Post}}(t) + V_j^+(t+1)\right)$$

Pure endowment

Thiele:

$$V_i^+(t) = a_i^{\text{Pre}}(t) + \sum_j v_t p_{ij}(t, t+1)\left(a_{ij}^{\text{Post}}(t) + V_j^+(t+1)\right)$$

Only two states: $\mathcal{S} = \{*, \dagger\}$ and $V_\dagger^+ \equiv 0$, hence we only have one function $V_*^+(t)$. Moreover all a_{ij}^{Post} are 0. The equation reduces to:

Pure endowment

Thiele:

$$V_i^+(t) = a_i^{\text{Pre}}(t) + \sum_j v_t p_{ij}(t, t+1)\left(a_{ij}^{\text{Post}}(t) + V_j^+(t+1)\right)$$

Only two states: $\mathcal{S} = \{*, \dagger\}$ and $V_\dagger^+ \equiv 0$, hence we only have one function $V_*^+(t)$. Moreover all a_{ij}^{Post} are 0. The equation reduces to:

$$V_*^+(t) = v_t p_{**}(t, t+1) V_*^+(t+1)$$

Pure endowment

Thiele:

$$V_i^+(t) = a_i^{\text{Pre}}(t) + \sum_j v_t p_{ij}(t, t+1) \left(a_{ij}^{\text{Post}}(t) + V_j^+(t+1) \right)$$

Only two states: $\mathcal{S} = \{*, \dagger\}$ and $V_\dagger^+ \equiv 0$, hence we only have one function $V_*^+(t)$. Moreover all a_{ij}^{Post} are 0. The equation reduces to:

$$V_*^+(t) = v_t p_{**}(t, t+1) V_*^+(t+1)$$

Moreover, the final condition is given as $V_*^+(T) = E$ where E is the survival benefit to be paid out at time T in case of survival.

Pure endowment

Thiele:

$$V_i^+(t) = a_i^{\text{Pre}}(t) + \sum_j v_t p_{ij}(t, t+1)\left(a_{ij}^{\text{Post}}(t) + V_j^+(t+1)\right)$$

Only two states: $\mathcal{S} = \{*, \dagger\}$ and $V_\dagger^+ \equiv 0$, hence we only have one function $V_*^+(t)$. Moreover all a_{ij}^{Post} are 0. The equation reduces to:

$$V_*^+(t) = v_t p_{**}(t, t+1) V_*^+(t+1)$$

Moreover, the final condition is given as $V_*^+(T) = E$ where E is the survival benefit to be paid out at time T in case of survival.

Procedure:

1. From $t = T$ you have $V_*^+(T) = E$.
2. Then go one step back to $t = T - 1$, to compute $V_*^+(T-1)$ you apply the recursion:
$$V_*^+(T-1) = v_{T-1} p_{**}(T-1, T) V_*^+(T).$$
3. Now you have $V_*^+(T-1)$ you can go on to computing $V_*^+(T-2)$ and so on down to $V_*^+(0)$.

Term insurance

Thiele:

$$V_i^+(t) = a_i^{\text{Pre}}(t) + \sum_j v_t p_{ij}(t, t+1)\left(a_{ij}^{\text{Post}}(t) + V_j^+(t+1)\right)$$

Term insurance

Thiele:

$$V_i^+(t) = a_i^{\text{Pre}}(t) + \sum_j v_t p_{ij}(t, t+1)\left(a_{ij}^{\text{Post}}(t) + V_j^+(t+1)\right)$$

Only two states: $\mathcal{S} = \{*, \dagger\}$ and $V_\dagger^+ \equiv 0$, hence we only have one function $V_*^+(t)$. Moreover a_i^{Pre} is 0 and $a_{*\dagger}^{\text{Post}} = B$. The equation reduces to:

Term insurance

Thiele:

$$V_i^+(t) = a_i^{\text{Pre}}(t) + \sum_j v_t p_{ij}(t, t+1)\left(a_{ij}^{\text{Post}}(t) + V_j^+(t+1)\right)$$

Only two states: $\mathcal{S} = \{*, \dagger\}$ and $V_\dagger^+ \equiv 0$, hence we only have one function $V_*^+(t)$. Moreover a_i^{Pre} is 0 and $a_{*\dagger}^{\text{Post}} = B$. The equation reduces to:

$$V_*^+(t) = v_t p_{**}(t, t+1) V_*^+(t+1) + v_t p_{*\dagger}(t, t+1) B$$

Term insurance

Thiele:

$$V_i^+(t) = a_i^{\text{Pre}}(t) + \sum_j v_t p_{ij}(t, t+1)\left(a_{ij}^{\text{Post}}(t) + V_j^+(t+1)\right)$$

Only two states: $\mathcal{S} = \{*, \dagger\}$ and $V_\dagger^+ \equiv 0$, hence we only have one function $V_*^+(t)$. Moreover a_i^{Pre} is 0 and $a_{*\dagger}^{\text{Post}} = B$. The equation reduces to:

$$V_*^+(t) = v_t p_{**}(t, t+1) V_*^+(t+1) + v_t p_{*\dagger}(t, t+1) B$$

Moreover, the final condition is given as $V_*^+(T) = 0$.

Term insurance

Thiele:

$$V_i^+(t) = a_i^{\text{Pre}}(t) + \sum_j v_t p_{ij}(t, t+1)\left(a_{ij}^{\text{Post}}(t) + V_j^+(t+1)\right)$$

Only two states: $\mathcal{S} = \{*, \dagger\}$ and $V_\dagger^+ \equiv 0$, hence we only have one function $V_*^+(t)$. Moreover a_i^{Pre} is 0 and $a_{*\dagger}^{\text{Post}} = B$. The equation reduces to:

$$V_*^+(t) = v_t p_{**}(t, t+1) V_*^+(t+1) + v_t p_{*\dagger}(t, t+1) B$$

Moreover, the final condition is given as $V_*^+(T) = 0$.

Procedure:

1. From $t = T$ you have $V_*^+(T) = 0$.
2. Then go one step back to $t = T-1$, to compute $V_*^+(T-1)$ you apply the recursion:

$$\begin{aligned} V_*^+(T-1) &= v_{T-1} p_{**}(T-1, T) \underbrace{V_*^+(T)}_{=0} + v_{T-1} p_{*\dagger}(T-1, T) B \\ &= v_{T-1} p_{*\dagger}(T-1, T) B. \end{aligned}$$

3. Now you have $V_*^+(T-1)$ you can go on to computing $V_*^+(T-2)$ and so on down to $V_*^+(0)$.

Exercise:

Write down Thiele's difference equation for the endowment (pure + term) insurance, pension and disability insurance.

UiO : Department of Mathematics
University of Oslo

David R. Banos

Life Insurance and Finance

Lecture 9: Thiele's difference equation