UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Constituent exam in: STK4500/9500 — Life Insurance and Finance

Day of examination: 10th June 2024

Examination hours: 3:00 pm – 7:00 pm

This problem set consists of 14 pages.

Appendices: None

Permitted aids: Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

This exam consists of four problems. The first one is of theoretical nature, while the three remaining problems are more applied. Make sure to be **precise** and **rigorous** when stating mathematical results and formulae. Please, **write clearly**, **orderly** and avoid scribbling.

Grading: The total score is 10 points. The grading scale is F [0,4), E [4,5), D [5, 6), C [6, 7), B [7, 8.5), A [8.5,10].

Problem 1 Theory (2 points)

Answer the following theoretical questions in the time setting you prefer: discrete or continuous.

(a) (0.5p) Define the concepts of present value, retrospective value and prospective value of a cash flow and explain what they mean.

Solution: We choose to display a solution in continuous time. Discrete time is fairly similar by, somehow, «exchanging integrals by sums».

We recall that a cash flow is a stochastic process C with a.s. càdlàg and bounded variation sample paths. Given a cash flow C, the present, retrospective and prospective values (at any time t) of the cash flow C are defined by

$$V(t,C) = \frac{1}{v(t)} \int_{[0,\infty)} v(s) dC(s),$$

$$V^{-}(t,C) = \frac{1}{v(t)} \int_{[0,t]} v(s) dC(s),$$

$$V^{+}(t,C) = \frac{1}{v(t)} \int_{(t,\infty)} v(s) dC(s),$$

(Continued on page 2.)

where $v(t)$ is a discount factor (today's value of one monetary unit to be exercised at time t) and the integrals are understood in the sense of Riemann-Stieltjes.

The cash flow C represents the balance of money fluctuations, hence $dC(s)$ is an instantaneous change of cash. The quantity $v(s)dC(s)$ is then today's value of such change of cash. Summing up, we obtain today's worthiness of the entire cash flow C, i.e.

$$\int_{[0,\infty)} v(s)dC(s)$$

is today's value of the entire cash flow. By multiplying by $\frac{1}{v(t)}$ we simply translate its value to an arbitrary future time t.

The quantities $V^-(t,C)$ and $V^+(t,C)$ have similar interpretation. The quantity $V^-(t,C)$ is then the t-value of the cash flow, but only tanking into account what has happened before t, i.e. $[0,t]$ and $V^+(t,C)$ is the t-value of the cash flow, but only tanking into account what will happen from t, i.e. (t,∞). Thus, $V(t,C) = V^-(t,C) + V^+(t,C)$.

Points: It does not need to be so detailed. Definition of cash flow is not needed either. Give 0.25 for present, retrospective and prospective and 0.25 for a meaningful interpretation. Subtract 0.1 for minor mistakes and 0.2 for major mistakes.

(b) (0.5p) Define the concept of policy functions and policy cash flow and give an interpretation. Note: remember to define and explain all the elements and notations you use.

Solution:

Policy functions $a_i(t)$ model accumulated payments for sojourns in state i and $a_{ij}(t)$ are punctual payments triggered by transitions from state i to state j.

Let Z be the (continuous time) Markov process with state space $\mathcal{Z}$ modelling the states of the insured. Define $I_i^Z(t) = \mathbb{I}_{\{Z(t)=i\}}$ and $N_{ij}^Z(t) = \#\{s \in [0,t] : Z(s-) = i, Z(s) = j\}$ where # denotes the counting measure. Then the policy cash flow determined by the policy functions has dynamics given by

$$dC(s) = \sum_{i\in\mathcal{Z}} I_i^Z(s)da_i(s) + \sum_{\substack{i,j\in\mathcal{Z}\\ j\neq i}} a_{ij}(s)dN_{ij}^Z(s).$$

The instantaneous change in cash $dC(s)$ is due to an instantaneous change in cash $da_i(s)$ as long as the insured is in state i, that is why we multiply by $I_i^Z(s)$, and due to payments a_{ij} for transitions from i to j. Here, $dN_{ij}^Z(s)$ is either 0 or 1 according to whether there is a transition from i to j or not. Then we sum over all possible states.

(Continued on page 3.)

Points: Give 0.3 for the formulas and 0.2 for the interpretation (0.1 for interpretation of policy functions and 0.1 for policy cash flow). Subtract 0.1 for minor mistakes and 0.2 for major mistakes.

(c) (1p) Explain the concept of actuarial reserve in mathematical terms.

Solution:

An actuarial reserve in colloquial terms is how much money the insurer needs to put aside to meet its contractual obligations. This is a very general definition. In mathematical terms we have the loss distribution, which is the distribution of possible payments the insured may trigger by contract. For instance, at time t, the remaining payments are given by $V^+(t, C)$ where C is the policy cash flow. Since C is stochastic so is $V^+(t, C)$. The quantity $V_i^+(t, C) = \mathbb{E}[V^+(t, C)|Z(t) = i]$ known as the expected prospective value is a theoretical possibility, although it is a bad one, since observations may differ a lot from the expected values of the distribution. A standard choice according to Solvency II regulation is to reserve the 99.5%-quantile of the loss distribution, i.e. the capital u such that

$$\mathbb{P}[V^+(t, C) > u] = 0.005\,.$$

The capital u is known as actuarial reserve.

Points: Give 0.5 for the expected prospective value and 0.5 for the quantile. Give special consideration to those who mention that expected values are bad solvency capitals. Subtract 0.1 for minor mistakes and 0.2 for major mistakes.

Problem 2 Permanent disability (2 points)

Consider the permanent disability model with states $\{*, \diamond, \dagger\}$ and (constant) transition rates $\mu_{*\diamond}, \mu_{*\dagger}, \mu_{\diamond\dagger}$ ($\mu_{\diamond *} = 0$). Constant transition rates are not realistic but allow for explicit computations. Hint: You may need that the solution of the differential equation $x'(t) = a(t)x(t) + b(t)$, $x(s) = x_0$, $s \geq t \geq 0$ is given by $x(t) = e^{-\int_t^s a(u)du}\left(x_0 - \int_t^s b(u)e^{\int_u^s a(v)dv}du\right)$.

(a) Find explicit expressions for the transition probabilities $p_{**}(t, s)$, $p_{\diamond\diamond}(t, s)$ and $p_{*\diamond}(t, s)$, $t < s$.

Solution: The probabilities p_{**} and $p_{\diamond\diamond}$ are fairly easy to compute if one knows the result from the lecture that says that

$$p_{ii}(t, s) = e^{-\sum_{j \neq i} \int_t^s \mu_{ij}(u)du},$$

whenever you cannot return to i if you depart, which is exactly the case in the permanent disability model (if you leave $*$ or $\diamond$ then you cannot return).

(Continued on page 4.)

Hence,

$$p_{**}(t,s) = e^{-(\mu_{*\diamond}+\mu_{*\dagger})(s-t)}, \quad p_{\diamond\diamond}(t,s) = e^{-\mu_{\diamond\dagger}(s-t)}, \quad t < s.$$

Recall Kolmogorov's backward equation:

$$\frac{d}{dt} p_{ij}(t,s) = \sum_{k \in \mathcal{Z}} \mu_{ik}(t) p_{kj}(t,s), \quad p_{ij}(s,s) = 1_{\{i=j\}},$$

where $\mu_{ii} = -\mu_i$.

In our case $i = *$ and $j = \diamond$ then

$$\frac{d}{dt} p_{*\diamond}(t,s) = -\mu_{**}(t) p_{*\diamond}(t,s) - \mu_{*\diamond}(t) p_{\diamond\diamond}(t,s).$$

We have that $\mu_{**} = -(\mu_{*\diamond} + \mu_{*\dagger})$ hence

$$\frac{d}{dt} p_{*\diamond}(t,s) = (\mu_{*\diamond} + \mu_{*\dagger}) p_{*\diamond}(t,s) - \mu_{*\diamond} p_{\diamond\diamond}(t,s).$$

Applying the hint with $x(t) = p_{*\diamond}(t,s)$, $a(t) = \mu_{*\diamond} + \mu_{*\dagger}$, $b(t) = -\mu_{*\diamond} p_{\diamond\diamond}(t,s)$ and $x_0 = 0$ we have

$$p_{*\diamond}(t,s) = \mu_{*\diamond} e^{-(\mu_{*\diamond}+\mu_{*\dagger})(s-t)} \int_t^s \underbrace{p_{\diamond\diamond}(u,s)}_{=e^{-\mu_{\diamond\dagger}(s-u)}} e^{(\mu_{*\diamond}+\mu_{*\dagger})(s-u)} du.$$

Merging the exponentials in the integral and integrating we obtain

$$p_{*\diamond}(t,s) = \frac{\mu_{*\diamond}}{\mu_{*\diamond} + \mu_{*\dagger} - \mu_{\diamond\dagger}} \left[e^{-\mu_{\diamond\dagger}(s-t)} - e^{-(\mu_{*\diamond}+\mu_{*\dagger})(s-t)} \right].$$

Points: Give 0.25 for writing down any of Kolmogorov's equation. Give 0.5 for writing down Kolmogorov equation for the specific case. Give 0.75 if there is an integral expression of the solution and give 1 if it is solved explicitly. Subtract 0.2 for minor mistakes and 0.5 for major mistakes.

(b) Consider an insurance policy with (constant) technical interest rate $r = 3\%$, maturity time $T = 10$ and age of the insured $z = 50$ years. Assume $\mu_{*\dagger} = \mu_{\diamond\dagger} = 0.003$ and $\mu_{*\diamond} = 0.01$. Compute the single premium of a disability policy paying a yearly disability pension of $D = 200\,000$ Norwegian kroner.

Solution:

Alternative I (using (a)): The only policy function for this insurance is given by

$$a_\diamond(t) = \begin{cases} Dt, & t \in [0,T], \\ DT, & t \in [T, \infty) \end{cases}.$$

(Continued on page 5.)

This function is continuous and a.e. differentiable with $a'_\diamond(t) = D$ on $(0, T)$. The expected prospective value, given that the insured is active at time t, is

$$V_*^+(t) = D\frac{1}{v(t)}\int_t^T v(s)p_{*\diamond}(z+t, z+s)ds, \quad t \in [0, T],$$

where $v(t) = e^{-rt}$ denotes the discount factor.

Using (a) and integrating we obtain

$$\begin{aligned} V_*^+(t) = \frac{\mu_{*\diamond}}{\mu_{*\diamond} + \mu_{*\dagger} - \mu_{\diamond\dagger}} D \Bigg[&\frac{1}{r + \mu_{\diamond\dagger}} \left(1 - e^{-(r+\mu_{\diamond\dagger})(T-t)}\right) \\ &- \frac{1}{r + \mu_{*\diamond} + \mu_{*\dagger}} \left(1 - e^{-(r+\mu_{*\diamond}+\mu_{*\dagger})(T-t)}\right) \Bigg]. \end{aligned}$$

Since $\Delta a_\diamond(0) = 0$ and $\frac{\mu_{*\diamond}}{\mu_{*\diamond}+\mu_{*\dagger}-\mu_{\diamond\dagger}} = 1$ the single premium at the beginning of the contract is

$$\begin{aligned} \pi_0 &= \Delta a_*(0) + V_*^+(0) \\ &= D\left[\frac{1}{r + \mu_{\diamond\dagger}}\left(1 - e^{-(r+\mu_{\diamond\dagger})T}\right) - \frac{1}{r + \mu_{*\diamond} + \mu_{*\dagger}}\left(1 - e^{-(r+\mu_{*\diamond}+\mu_{*\dagger})T}\right)\right] \\ &= 200\,000\Big[\frac{1}{0.03 + 0.003}\left(1 - e^{-(0.03+0.003)\cdot 10}\right) \\ &\quad - \frac{1}{0.03 + 0.01 + 0.003}\left(1 - e^{-(0.03+0.01+0.003)\cdot 10}\right)\Big] \\ &\approx 77\,953.43\,. \end{aligned}$$

Alternative II (without (a)): If the candidate could not solve Kolmogorov equation, there is an "independent" way to solve the problem: Thiele's equation.

We omit the symbol $+$ in the expected prospective values V_i^+ to ease notation. For the three-state model we have that Thiele's equation is given by

$$\begin{aligned} V'_*(t) &= rV_*(t) - \sum_{j\neq *} \mu_{*j}\left(V_j(t) - V_*(t)\right), \\ V'_\diamond(t) &= rV_\diamond(t) - a_\diamond(t) - \sum_{j\neq\diamond} \mu_{\diamond j}\left(V_j(t) - V_\diamond(t)\right), \end{aligned}$$

with $V_*(T-) = V_\diamond(T-) = 0$.

Simplifying we arrive to

$$\begin{aligned} V'_*(t) &= (r + \mu_{*\diamond} + \mu_{*\dagger})V_*(t) - \mu_{*\diamond}V_\diamond(t), \\ V'_\diamond(t) &= (r + \mu_{\diamond\dagger})V_\diamond(t) - D, \end{aligned}$$

(Continued on page 6.)

with $V_*(T-) = V_\diamond(T-) = 0$.

Now, to find $V_\diamond^+$ we use the hint with $a(t) = r + \mu_{\diamond\dagger}$ and $b(t) = -D$. Thus, we obtain

$$V_\diamond^+(t) = D\frac{1}{r + \mu_{\diamond\dagger}}\left(1 - e^{-(r+\mu_{\diamond\dagger})(T-t)}\right).$$

Finally, to find V_*^+ we use the hint with $a(t) = r + \mu_{*\diamond} + \mu_{*\dagger}$ and $b(t) = -\mu_{*\diamond}V_\diamond(t)$. Hence,

$$V_*^+(t) = e^{-(r+\mu_{*\diamond}+\mu_{*\dagger})(T-t)}\mu_{*\diamond}\int_t^T V_\diamond^+(u)e^{(r+\mu_{*\diamond}+\mu_{*\dagger})(T-u)}du.$$

Since we are interested in the single premium we just need the value at $t = 0$, thus

$$\pi_0 = \Delta a_*(0) + V_*^+(0) = e^{-(r+\mu_{*\diamond}+\mu_{*\dagger})T}\mu_{*\diamond}\int_0^T V_\diamond^+(u)e^{(r+\mu_{*\diamond}+\mu_{*\dagger})(T-u)}du.$$

The final expression we obtain by integrating is

$$\begin{aligned} V_*^+(t) = \frac{\mu_{*\diamond}D}{r+\mu_{\diamond\dagger}}\Bigg[&\frac{1}{r+\mu_{*\diamond}+\mu_{*\dagger}}\left(1 - e^{(r+\mu_{*\diamond}+\mu_{*\dagger})(T-t)}\right) \\ &- \frac{1}{\mu_{*\diamond}+\mu_{*\dagger}+\mu_{\diamond\dagger}}\left(e^{-(r+\mu_{*\diamond}+\mu_{*\dagger})(T-t)} - e^{-(r+\mu_{\diamond\dagger})(T-t)}\right). \end{aligned}$$

Evaluating at $t = 0$ and inserting the values yields obviously the same result as in alternative I.

Points: Any method counts equally. Alt I: give 0.5 if there is a final expression that is not computed. Alt II: give 0.5 if Thiele is stated and 0.75 if it is solved but the candidate does not arrive at the numerical solution. Subtract 0.2 for minor mistakes and 0.5 for major mistakes.

Problem 3 Solvency capital requirement (4 points)

Consider a life insurance, in continuous time, that pays benefit B in case of death before expiration of the contract or a benefit E in case of survival at expiration time. As usual, denote by Z the continuous time Markov process with state space $\mathcal{Z} = \{*, \dagger\}$ modelling the states of the insured. Denote by τ the death time of the insured. Mortality is a deterministic time-dependent function $\mu(t)$ such that $\lim_{t\to\infty}\int_0^t \mu(u)du = \infty$.

The specifications of the policy are: Expiration date T years from now, insured is z years old at inception, technical discount factor $v(t)$ stands for today's value of one monetary unit to be paid out at time $t \geq 0$.

(Continued on page 7.)

(a) Show that the conditional density function of $\tau|\tau > t$ is given by

$$f_{\tau|\tau>t}(s|t) = \mu(z+s)e^{-\int_t^s \mu(z+u)du}, \quad s \geq t.$$

and that of $\tau|\tau \leq t$ is given by

$$f_{\tau|\tau\leq t}(s|t) = \frac{\mu(z+s)e^{-\int_0^s \mu(z+u)du}}{1-e^{-\int_0^t \mu(z+u)du}}, \quad 0 \leq s \leq t.$$

Solution:

As always, Z is the continuous time Markov process modelling the states $\mathcal{Z}$ of the insured. Here, $\mathcal{Z} = \{*, \dagger\}$. Since τ has no atoms, the event $\{\tau > t\}$ is equivalent to $\{Z(t) = *\}$. The variable τ is the «age» of the contract, so at time τ the insured is $z+\tau$ years old. We thus have,

$$\mathbb{P}[\tau > s|\tau > t] = p_{**}(z+t, z+s) = e^{-\int_t^s \mu(z+u)du}, \quad s > t.$$

The distribution function of $\tau|\tau > t$ is

$$F_{\tau|\tau>t}(s|t) = 1 - e^{-\int_t^s \mu(z+u)du}$$

and the density function is obtained by differentiating, that is

$$f_{\tau|\tau>t}(s|t) = \mu(z+s)e^{-\int_t^s \mu(z+u)du}, \quad s \in [t, \infty).$$

Now for $s > t$ we have $\mathbb{P}[\tau \leq s|\tau \leq t] = 1$ and for $s \in [0, t]$ we have,

$$F_{\tau|\tau\leq t}(s|t) = \mathbb{P}[\tau \leq s|\tau \leq t] = \frac{\mathbb{P}[\tau \leq s]}{\mathbb{P}[\tau \leq t]}.$$

Differentiating and using the previous information we have

$$f_{\tau|\tau\leq t}(s|t) = \frac{\mu(z+s)e^{-\int_0^s \mu(z+u)du}}{1-e^{-\int_0^t \mu(z+u)du}}, \quad s \in [0, t].$$

Points: Give 0.5 for each. Give 0.2 for arguing the connection between death time and survival probability. Subtract 0.2 for minor mistakes and 0.5 for major mistakes.

(b) Let C denote the policy cash flow. Prove that for every $t \in [0, T]$, the retrospective value $V^-(t, C)$ and the prospective value $V^+(t, C)$ of this policy are given by

$$V^-(t, C) = E\mathbb{I}_{\{T\}}(t)\mathbb{I}_{(T,\infty)}(\tau) + B\frac{v(\tau)}{v(t)}\mathbb{I}_{[0,t]}(\tau),$$

$$V^+(t, C) = E\frac{v(T)}{v(t)}\mathbb{I}_{(T,\infty)}(\tau)\mathbb{I}_{[0,T)}(t) + B\frac{v(\tau)}{v(t)}\mathbb{I}_{(t,T]}(\tau).$$

(Continued on page 8.)

Solution:

The policy cash flow is given by

$$dC(s) = I_*^Z(s)da_*(s) + a_{*\dagger}(s)dN_{*\dagger}^z(s),$$

where the policy functions are given by

$$a_*(s) = \begin{cases} E, & s \geq T, \\ 0, & s \in [0,T], \end{cases} \quad , \quad a_{*\dagger}(s) = \begin{cases} B, & t \in [0,T], \\ 0, & \text{otherwise.} \end{cases} \quad .$$

The function a_* is a.e. constant with jump $\Delta a_*(T) = E$. The random variables $I_*^Z(s)$ and $N_{*\dagger}^Z(s)$ can be expressed as $I_*^Z(s) = \mathbb{I}_{[0,\tau)}(s)$ and $N_{*\dagger}^Z(s) = \mathbb{I}_{[\tau,\infty)}(s)$.

The retrospective value is the Riemann-Stieltjes integral given by

$$\begin{aligned} V^-(t,C) &= \frac{1}{v(t)}E\int_{[0,t]} v(s)dC(s) \\ &= E\frac{1}{v(t)}\int_{[0,\infty)} v(s)\mathbb{I}_{[0,t]}(s)\mathbb{I}_{[0,\tau)}(s)d\mathbb{I}_{[T,\infty)}(s) \\ &\quad + \frac{1}{v(t)}\int_{[0,\infty)} v(s)B\mathbb{I}_{[0,T]}(s)\mathbb{I}_{[0,t]}(s)d\mathbb{I}_{[\tau,\infty)}(s). \end{aligned}$$

The jump of the second integral occurs at time τ. Therefore,

$$V^-(t,C) = E\frac{v(T)}{v(t)}\mathbb{I}_{[0,t]}(T)\mathbb{I}_{[0,\tau)}(T) + B\frac{v(\tau)}{v(t)}\mathbb{I}_{[0,T]}(\tau)\mathbb{I}_{[0,t]}(\tau).$$

Now, since $t \in [0,T]$ then $\mathbb{I}_{[0,t]}(T) = 1$ if, and only if $t = T$ among $t \in [0,T]$ and in such case $E\frac{v(T)}{v(t)} = E$. Moreover $[0,t] \subseteq [0,T]$ for $t \in [0,T]$ we have

$$V^-(t,C) = E\mathbb{I}_{\{T\}}(t)\mathbb{I}_{[T,\infty)}(\tau) + B\frac{v(\tau)}{v(t)}\mathbb{I}_{[0,t]}(\tau).$$

For the prospective value we do the same calculations but with integrand $\mathbb{I}_{(t,\infty)}(s)$ instead.

$$\begin{aligned} V^+(t,C) &= \frac{1}{v(t)}E\int_{[0,t]} v(s)dC(s) \\ &= E\frac{1}{v(t)}\int_{[0,\infty)} v(s)\mathbb{I}_{(t,\infty)}(s)\mathbb{I}_{[0,\tau)}(s)d\mathbb{I}_{[T,\infty)}(s) \\ &\quad + \frac{1}{v(t)}\int_{[0,\infty)} v(s)B\mathbb{I}_{[0,T]}(s)\mathbb{I}_{(t,\infty)}(s)d\mathbb{I}_{[\tau,\infty)}(s) \\ &= E\frac{v(T)}{v(t)}\mathbb{I}_{(t,\infty)}(T)\mathbb{I}_{[0,\tau)}(T) + B\frac{v(\tau)}{v(t)}\mathbb{I}_{[0,T]}(\tau)\mathbb{I}_{(t,\infty)}(\tau) \end{aligned}$$

(Continued on page 9.)

and the result follows by simplification of indicators.

The retrospective value tells us the (stochastic) t-value of the passed cash, depending on whether death has happened before of after t and so on. Similarly, the prospective value tells us the (stochastic) t-value of the future possible payments depending on the future states.

Points: Give 0.5 for each one. Give 0.2 if the candidate writes down both theoretical definitions of values. Give 0.5 if the candidate writes down the policy functions, cash flow and definitions of values. Give 0.2 for an interpretation. Subtract 0.2 for minor mistakes and 0.5 for major mistakes.

Assume, in the rest of the exercise, that mortality is a constant $\mu > 0$. This is not a realistic assumption but it allows for explicit computations. Moreover, let r be a constant technical interest rate, hence from now on $v(t) = e^{-rt}$.

(c) Show that the *expected* retrospective values of this insurance are given by

$$V_*^-(t, C) = E\mathbb{I}_{\{T\}}(t), \quad t \in [0, T],$$

and

$$V_\dagger^-(t, C) = e^{rt} \frac{B\mu}{r+\mu} \frac{1 - e^{-(r+\mu)t}}{1 - e^{-\mu t}}, \quad t \in [0, T].$$

and give an interpretation of these quantities.

Solution: For the state $i = *$ we have that the event $\{Z(t) = *\}$ is equivalent to $\{\tau > t\}$ an hence the second term of $V^-(t, C)$ drops out. Thus, we simply use the distribution of $\tau|\tau > t$ to compute the conditional expectation.

$$\begin{aligned}
V_*^-(t, V) &= \mathbb{E}[V^-(t, C)|Z(t) = *] \\
&= E\mathbb{I}_{\{T\}}(t)\mathbb{E}[\mathbb{I}_{(T,\infty)}(\tau)|Z(t) = *]\mathbb{I}_{\{T\}}(t) \\
&= E\mathbb{I}_{\{T\}}(t)\mathbb{P}[\tau > T|\tau > t] \\
&= E\mathbb{I}_{\{T\}}(t)e^{-\int_t^T \mu(z+u)du} \\
&= E\mathbb{I}_{\{T\}}(t).
\end{aligned}$$

The above value is the t-value of what we *should* expect to have paid out to the insured at time t if the insured is alive at time t. But in such case, there will not be any death benefit and the survival benefit will only be paid at time $t = T$, hence $V_*^-(t, C) = 0$ for all $t \in [0, T)$. Finally and obviously $V_*^-(T, C) = E$.

Further, for $i = \dagger$ we have $\{Z(t) = \dagger\}$ is equivalent to $\{\tau \le t\}$ and thus the first term in $V^-(t, C)$ drops since survival benefit will not be paid. Then,

$$V_\dagger^-(t, C) = B\mathbb{E}\left[\frac{v(\tau)}{v(t)}\mathbb{I}_{[0,t]}(\tau)\middle|\tau \le t\right].$$

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We need to use the conditional distribution of $\tau|\tau \le t$ from (a) to compute the expectation. That is,

$$V_{\dagger}^{-}(t,C) = \frac{B}{v(t)} \int_0^t v(s) f_{\tau|\tau\le t}(s|t) ds = \frac{B}{v(t)} \int_0^t v(s) \frac{\mu e^{-\mu s}}{1-e^{-\mu t}} ds,$$

and the result follows by computing the (almost immediate) integral.

Points: Give 0.5 for each and 0.2 for only an interpretation. Give 0.3 for each if there is a written expression that is not solved completely. Subtract 0.2 for minor and 0.5 for serious mistakes.

(d) The *solvency capital requirement* at the beginning of the contract is defined as the value u_α for which

$$\mathbb{P}[V^+(0,C) > u_\alpha] = \alpha,$$

for a suitably small α, usually $\alpha = 0.005$.

Find a formula for the solvency capital requirement u_α that the insurer needs to keep aside to stay solvent in terms of all the parameters. You can assume that $Ee^{-rT} < Be^{-rT} < u_\alpha < B$ if you want, since a different assumption is not so meaningful in real life.

Solution:

Recall from (b) that

$$V^+(0,C) = Ev(T)\mathbb{I}_{(T,\infty)}(\tau) + Bv(\tau)\mathbb{I}_{(0,T]}(\tau).$$

We will use the law of total probability, i.e. $\mathbb{P}[A|B] = \mathbb{P}[A|B]\mathbb{P}[B] + \mathbb{P}[A|B^c]\mathbb{P}[B^c]$ for any events A, B such that $\mathbb{P}[B] \neq 0$.

Thus,

$$\begin{aligned}\mathbb{P}[V^+(0,C) > u] =& \mathbb{P}[V^+(0,C) > u|\tau > T]\mathbb{P}[\tau > T] \\ &+ \mathbb{P}[V^+(0,C) > u|\tau \le T]\mathbb{P}[\tau \le T].\end{aligned}$$

Hence, if $\tau > T$ then

$$V^+(0,C) = Ev(T) \quad \text{on} \quad \{\tau > T\},$$

and if $\tau \le T$ then

$$V^+(0,C) = Bv(\tau) \quad \text{on} \quad \{\tau \le T\}.$$

Therefore,

$$\begin{aligned}\mathbb{P}[V^+(0,C) > u] =& \mathbb{P}[Ev(T) > u|\tau > T]\mathbb{P}[\tau > T] \\ &+ \mathbb{P}[Bv(\tau) > u|\tau \le T]\mathbb{P}[\tau \le T] \\ =& \mathbb{I}_{\{Ev(T)>u\}}\mathbb{P}[\tau > T] \\ &+ \mathbb{P}\left[\tau < -\frac{1}{r}\log\frac{u}{B}\middle|\tau \le T\right]\mathbb{P}[\tau \le T]\end{aligned}$$

(Continued on page 11.)

Now, $Ev(T) < u$ so the first probability is zero. We also have that $0 < -\frac{1}{r}\log\frac{u}{B} < T$. Indeed, the latter is true if, and only if

$$0 > \log\frac{u}{B} > -rT.$$

Taking exponentials

$$1 > \frac{u}{B} > e^{-rT},$$

if, and only if

$$Be^{-rT} < u < B$$

which is assumed. Hence,

$$\mathbb{P}\left[\tau < -\frac{1}{r}\log\frac{u}{B}\Big|\tau \le T\right] = F_{\tau|\tau\le T}\left(-\frac{1}{r}\log\frac{u}{B}\Big|T\right) = \frac{\mathbb{P}\left[\tau \le -\frac{1}{r}\log\frac{u}{B}\right]}{\mathbb{P}[\tau \le T]}.$$

Altogether,

$$\mathbb{P}[V^+(0,C) > u] = 1 - e^{\frac{1}{r}\log\frac{u}{B}\cdot\mu} = 1 - \left(\frac{u}{B}\right)^{\mu/r}.$$

Hence, we need to choose u_α such that

$$u_\alpha \ge (1-\alpha)^{r/\mu}B.$$

We see that if we choose $u \ge B$ (the highest benefit that at worst case is paid out right after inception, then we are indeed solvent with probability one.

Points: Give 0.2 for using law of total probability and 0.5 for identifying $V^+(0,C)$ on each event. Subtract 0.2 for minor and 0.5 for serious mistakes.

Problem 4 Guaranteed inheritance (2 points)

A person wishes to invest C_0 monetary units into a mutual fund managed by an insurance company. The investment portfolio consists of a risk-free asset whose value at time t is given by $B(t)$ (a savings account) and a risky fund whose value at time t is given by $S(t)$. The company will allocate, at each time s, the proportion $\delta(s) \in [0,1]$ into the savings account and the rest into the mutual fund. The contract goes on forever and in case of the customer's demise, the company will automatically pay out the value of the portfolio at the death time to the family.

Let $v(t)$ denote the discount factor corresponding to the savings account with (constant) short rate r, i.e. you can assume that $B(t) = \frac{1}{v(t)}$.

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Moreover, let $p(z+t, z+s)$ and $\mu(z+s)$ denote, respectively, the survival probability and mortality of the customer of age z at inception.

Finally, assume that the fund S is modelled under the Black-Scholes model with constant rate r (same as $v(t)$) and constant volatility σ.

(a) Compute the expected prospective value of this insurance.

What is the single premium of this policy? Could you elaborate?

Solution:

The investment plan is to allocate $\delta(s)$ into B and the rest in to S out of the C_0 monetary units. The value of the investment will be paid out if a transition from $*$ to $\dagger$ occurs. Thus, the policy function of this unit-linked insurance is given by

$$h_{*\dagger}(s, S(s)) = C_0\delta(s)B(s) + \frac{C_0}{S(0)}(1-\delta(s))S(s), \quad s \geq 0.$$

The expected prospective value of this insurance is given by

$$V_*^+(t, S(t)) = \frac{1}{v(t)}\int_t^\infty v(s)p(z+t, z+s)\mu(z+s)\mathbb{E}_{\mathbb{Q}}[h_{*\dagger}(s, S(s))|\mathcal{F}_t]ds,$$

where $\mathbb{Q}$ is an equivalent martingale measure. Because of the latter fact, we have that $v(t)S(t)$ is a $\mathbb{Q}$-martingale. Therefore,

$$\begin{aligned}
v(s)\mathbb{E}_{\mathbb{Q}}[h_{*\dagger}(s, S(s))|\mathcal{F}_t] &= v(s)C_0\delta(s)B(s) + \frac{C_0}{S(0)}(1-\delta(s))\mathbb{E}_{\mathbb{Q}}[v(s)S(s)|\mathcal{F}_t] \\
&= C_0\delta(s) + \frac{C_0}{S(0)}(1-\delta(s))v(t)S(t).
\end{aligned}$$

Altogether, we have

$$V_*^+(t, S(t)) = \int_t^\infty \left(\frac{C_0}{v(t)}\delta(s) + \frac{C_0}{S(0)}(1-\delta(s))S(t)p(z+t, z+s)\mu(z+s)ds,\right.$$

Noting that $\frac{1}{v(t)} = B(t)$ and collecting terms that multiply $\delta(s)$ we get

$$\begin{aligned}
V_*^+(t, S(t)) =& C_0\left(B(t) - \frac{S(t)}{S(0)}\right)\int_t^\infty \delta(s)p(z+t, z+s)\mu(z+s)ds \\
&+ \frac{C_0}{S(0)}S(t)\int_t^\infty \mu(z+s)p(z+t, z+t)ds.
\end{aligned}$$

Finally, since

$$\int_t^\infty \mu(z+s)p(z+t, z+s)ds = -p(z+t, z+s)\Big|_{s=t}^{s=\infty} = 1,$$

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we have

$$V_*^+(t,S(t)) = C_0\left(B(t) - \frac{S(t)}{S(0)}\right)\int_t^\infty \delta(s)p(z+t,z+s)\mu(z+s)ds$$
$$+\frac{C_0}{S(0)}S(t).$$

The initial premium is found by inserting $t=0$ and thus,

$$\pi_0 = V_*^+(0,S(0)) = C_0.$$

This is obvious since this is the initial capital replicating the final investment. Since the benefit the policy has no expiration date and there is no guarantee, the insured simply collects the value of the portfolio at time of death, bearing thus all risk. The insurance company here is bearing no risk, just managing the portfolio. In real life, the insured would need to pay a fee for management costs.

Points: Give 0.5 if the candidate writes down the general formula. Give 0.2 if he candidate identifies the need to price under $\mathbb{Q}$. Give 0.2 for the explanation. Subtract 0.2 for minor and 0.5 for serious mistakes.

(b) Assume now, in addition, that the company offers the insured a guaranteed amount of G monetary units, meaning that, no matter what, the family will receive the guaranteed amount G as inheritance if the investments did not turn out to be at least as good as G. Compute the single premium again (you can leave it in terms of known formulas) and elaborate on the difference from item (a). Hint: manipulate the benefit in order to turn it into a call option with varying strike price plus some remaining.

Solution:

The death benefit changes from

$$h_{*\dagger}(s,S(s)) = C_0\delta(s)B(s) + \frac{C_0}{S(0)}(1-\delta(s))S(s), \quad s \geq 0,$$

to

$$h_{*\dagger}(s,S(s)) = \max\left\{C_0\delta(s)B(s) + \frac{C_0}{S(0)}(1-\delta(s))S(s), G\right\}, \quad s \geq 0.$$

It is a general fact that $\max\{A,G\} = (A-G)_+ + G$. Hence,

$$h_{*\dagger}(s,S(s)) = \left(C_0\delta(s)B(s) + \frac{C_0}{S(0)}(1-\delta(s))S(s) - G\right)_+ + G.$$

Then,

$$h_{*\dagger}(s,S(s)) = \frac{C_0}{S(0)}(1-\delta(s))\,(S(s)-K(s))_+ + G.$$

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$$K(s) \triangleq \frac{G - C_0\delta(s)B(s)}{\frac{C_0}{S(0)}(1-\delta(s))},$$

provided $\frac{C_0}{S(0)}(1-\delta(s)) > 0$. If not, the benefit is deterministic and we do not need to do all this reasoning.

We denote by $BS(t,s,S(t),K(s))$ the Black-Scholes price of a call option at time t with maturity s and strike price $K(s)$. That is

$$\frac{v(s)}{v(t)}\mathbb{E}_{\mathbb{Q}}\left[(S(s)-K(s))_+ \Big| \mathcal{F}_t\right] = BS(t,s,S(t),K(s)).$$

Then, the price for this polity at time t is

$$\begin{aligned} V_*^+(t,S(t)) &= \int_t^\infty \frac{v(s)}{v(t)}\mathbb{E}_{\mathbb{Q}}\left[h_{*+}(s,S(s))|\mathcal{F}_t\right]p(t,s)\mu(s)ds \\ &= \int_t^\infty \frac{C_0}{S(0)}(1-\delta(s))\left(BS(t,s,S(t),K(s)) + \frac{v(s)}{v(t)}G\right)p(t,s)\mu(s)ds \end{aligned}$$

The single premium is

$$\pi_0 = V_*^+(0,S(0)) = \int_0^\infty \frac{C_0}{S(0)}(1-\delta(s))\left(BS(0,s,S(0),K(s)) + v(s)G\right)p(0,s)\mu(s)ds$$

which is different from C_0. Here, the insurer is bearing some risk, namely the risk of having to pay a guarantee that is above the market price of the investment.

Points: Give 0.5 if the candidate writes down the general formula. Give 0.2 if he candidate identifies the need to price under $\mathbb{Q}$ and the relation to the Black-Scholes formula. Give 0.2 for the interpretation. Subtract 0.2 for minor and 0.5 for serious mistakes.

GOOD LUCK!

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