UiO : Department of Mathematics
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Mathematical Finance

STK4500 Life insurance and finance

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Preliminaries

In the previous chapter we studied how to construct processes of the form "integral of a process w.r.t. another process". That is

$$\int_0^t Y_s dX_s,$$

where X is a **semimartingale.** Semimartingales are a (rather big) class of processes that serve as "good integrators" in stochastic analysis.
Recall: a **semimartingale** is a process X that can be split into

$$X_t = X_0 + A_t + M_t,$$

where A is a càdlàg adapted process of bounded variation and M is a local-martingale.

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where A is a càdlàg adapted process of bounded variation and M is a local-martingale.
We will focus on the case where $X = B$ being B a **Brownian motion** (which is in fact a martingale).

Brownian motion

Definition (Brownian motion)

Let $B = \{B_t, t \geq 0\}$ be a stochastic process and $\mathcal{F}$ its natural filtration. Then B is a **Brownian motion** if it satisfies the following assumptions

1. $B_0 = 0$, $\mathbb{P}$-a.s.
2. B has independent increments, that is given $0 \leq t_0 \leq \cdots \leq t_n$, $n \geq 1$, $B_{t_i} - B_{t_{i-1}}$ is independent of $\mathcal{F}_{t_{i-1}}$.
3. B has stationary increments, that is $B_{t_i} - B_{t_{i-1}}$ has the same distribution as $B_{t_i - t_{i-1}}$ and they are normally distributed with mean zero and variance $t_i - t_{i-1}$.

As a consequence of Kolmogorov's continuity criterion one has that the sample paths $t \mapsto B_t$ are $\mathbb{P}$-a.s. **continuous**.

The following important result characterizes Brownian motion.

Theorem (Lévy's characterization theorem)

Let $B = \{B_t, t \in [0, T]\}$ be a stochastic process on $(\Omega, \mathcal{A}, \mathbb{P})$ and $\mathcal{F}$ its natural filtration. Then the following are equivalent:

1. *B is a Brownian motion.*
2. *B is an $(\mathcal{F}, \mathbb{P})$-martingale with $B_0 = 0$ $\mathbb{P}$-a.s. and quadratic variation $[B, B]_t = t$, $\mathbb{P}$-a.s.*

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In other words... **Brownian motion** is the only **martingale** (starting at 0) with **quadratic variation** t.

Theorem (Itô's formula w.r.t. Brownian motion)

Let $f \in C^{1,2}([0,T] \times \mathbb{R})$ *and* B *a Brownian motion. Then*

$$f(t, B_t) = f(0, B_0) + \int_0^t \left(\frac{\partial}{\partial s} f(s, B_s) + \frac{1}{2} \frac{\partial^2}{\partial x^2} f(s, B_s) \right) ds$$
$$+ \int_0^t \frac{\partial}{\partial x} f(s, B_s) dB_s.$$

Proof.

This is a consequence the general **Itô formula** from the previous chapter by taking $X = B$ and using that $\Delta B_t = 0$ and $[B, B]_t = t$. ∎

We now know how to construct integrals with ds (Lebesgue) and with dB_s (Itô). So, given two adapted processes u and v such that $\int_0^t \mathbb{E}[|u_s|]ds < \infty$ and $\int_0^t \mathbb{E}[|v_s|^2]ds < \infty$ we can look at processes of the form

$$X_t = X_0 + \int_0^t u_s ds + \int_0^t v_s dB_s.$$

Such processes are known as **Itô processes** and they are obviously semimartingales.

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If we choose u and v as two deterministic functions on $b, \sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ and define $u_s = b(s, X_s)$ and $v_s = \sigma(s, X_s)$ then we are looking at processes like

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s. \tag{1}$$

The process

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s \tag{2}$$

written in differential form would be:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad t \in [0, T].$$

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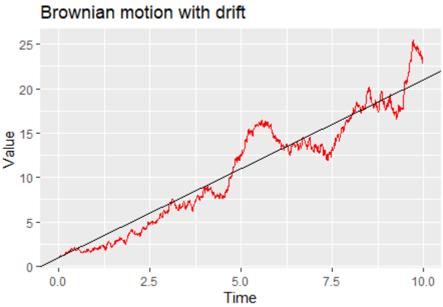
The above mathematical expression is known as a stochastic differential equation. It describes the dynamics of X (in differential form) and it has no other meaning than (2)
The part with dt is known as the **drift part (trend)** and the part with dB_t is known as the **diffusion part (volatility).**

Example (Brownian motion with drift)

If $b(t,x) = \mu$ and $\sigma(t,x) = \sigma > 0$ then $X_t = X_0 + \mu t + \sigma B_t, \quad t \in [0,T]$. This is one of the simplest models for modelling market movements. It is not very suitable since X can be negative. We have $\mathbb{E}[X_t] = \mu t, \quad \mathbb{V}ar[X_t] = \sigma^2 t$.



Example (Geometric Brownian motion)

If $b(t,x) = \mu x$ and $\sigma(t,x) = \sigma x$, $\sigma > 0$, then

$$dX_t = \mu X_t dt + \sigma X_t dB_t, \quad t \in [0,T]. \tag{3}$$

To solve the above SDE we start by "guessing" that the solution is of the form

$$X_t = X_0 \exp\left(\int_0^t \alpha(s)ds + \int_0^t \beta(s)dB_s\right), \quad t \in [0,T],$$

where α and β are adapted processes such that the integrals are well-defined.

Example (Geometric Brownian motion)

Let $Y_t \triangleq \int_0^t \alpha(s)ds + \int_0^t \beta(s)dB_s$ then

$$X_t = X_0 e^{Y_t}.$$

Applying Itô's formula for semimartingales (in this case Y) we have

$$\begin{aligned} dX_t &= X_t dY_t + \frac{1}{2} X_t d[Y,Y]_t \\ &= X_t \left(\alpha(t)dt + \beta(t)dB_t + \frac{1}{2}\beta^2(t)dt \right) \\ &= X_t \left(\alpha(t) + \frac{1}{2}\beta^2(t) \right) dt + X_t \beta(t) dB_t. \end{aligned}$$

Example (Geometric Brownian motion)

Comparing with (3) we have

$$\beta(t) = \sigma, \quad \alpha(t) + \frac{1}{2}\beta^2(t) = \mu$$

which gives $\alpha(t) = \mu - \frac{1}{2}\sigma^2$. Hence, the solution to (3) is given by

$$X_t = X_0 \exp\left(\left(\mu - \frac{1}{2}\right) t + \sigma B_t\right), \quad t \in [0, T].$$

The above process has been extensively used and, used as market price process, gives rise to the Black-Scholes formula when pricing European options. As you can see, X_t is log-normally distributed with

$$\mathbb{E}[X_t] = X_0 e^{\mu t}, \quad \mathbb{V}ar[X_t] = X_0^2 \left(e^{\sigma^2 t} - 1\right) e^{2\mu t}.$$

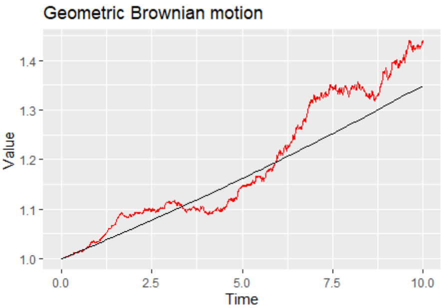


Figure: A sample path of a Geometric Brownian motion with $X_0 = 1$, $\mu = 0.03$, $\sigma = 0.001$.

Example (Ornstein-Uhlenbeck process)

If $b(t,x) = a(b-x)$ and $\sigma(t,x) = \sigma > 0$ then

$$dX_t = a(b - X_t)dt + \sigma dB_t, \quad t \in [0,T]. \tag{4}$$

As before, we try to guess how the solution may look like. Apply Itô's formula to $Y_t = e^{at}X_t$, by the product rule we have

$$\begin{aligned} dY_t &= d[e^{at}X_t] = d[e^{at}]X_t + e^{at}dX_t \\ &= ae^{at}X_t + e^{at}\left[a(b - X_t)dt + \sigma dB_t\right] \\ &= e^{at}\left[(aX_t + ab - aX_t)\,dt + \sigma dB_t\right] \\ &= e^{at}abdt + e^{at}\sigma dB_t. \end{aligned}$$

Example (Ornstein-Uhlenbeck process)

Hence, integrating both sides

$$e^{at}X_t - X_0 = ab\int_0^t e^{as}ds + \sigma\int_0^t e^{as}dB_s.$$

The finite variation part can be computes explicitly, then it follows that

$$X_t = X_0 e^{-at} + b(1-e^{-at}) + \sigma\int_0^t e^{-a(t-s)}dB_s, \quad t\ in[0,T].$$

It is readily seen that the above process is normally distributed with

$$\mathbb{E}[X_t] = X_0 e^{-at} + b(1-e^{-at}), \quad \mathbb{V}ar[X_t] = \frac{\sigma^2}{2a}\left(1-e^{-2at}\right).$$

Example (Ornstein-Uhlenbeck process)

The above process, when used for interest rate modelling, is known under the name **Vasicek.**

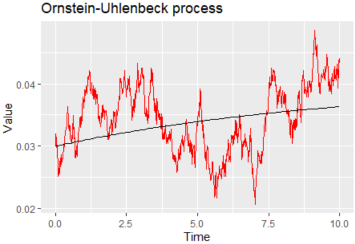


Figure: A sample path of an Ornstein-Uhlenbeck process with $X_0 = 0.03$, $a = 0.1$, $b = 0.04$, $\sigma = 0.001$.

Market model

In this section we assume that we have a rather simple market consisting of a **risky stock** or **fund** whose price at time t is given by S_t, and a **riskless fixed-income asset**, e.g. a back account given by B. The dynamics of B are

$$\frac{dB_t}{B_t} = r_t dt, \quad B_0 = 1, \quad t \in [0, T], \tag{5}$$

where r is a deterministic interest rate curve, and S has (semimartingale) dynamics described by

$$\frac{dS_t}{S_t} = \mu(t, S_t)dt + \sigma(t, S_t)dW_t, \quad S_0 > 0, \quad t \in [0, T], \tag{6}$$

where μ and σ are suitable deterministic functions in such a way that the above SDE is well-posed and a solution exists.

The process B is related to v from previous sections and it serves as discount factor. Actually, it is easy to see that

$$B_t = \exp\left(\int_0^t r_s ds\right) = \frac{1}{v(t)}, \quad t \in [0, T].$$

Trading strategies and value of the portfolio

Let $\eta^i = \{\eta^i_t, t \in [0,T]\}$, $i = 0, 1$ be two stochastic processes in $(\Omega, \mathcal{F}_T, \mathbb{P})$ and denote $\eta = (\eta^0, \eta^1)$. Here, η^0_t denotes the units invested in **money** by time t and η^1_t the units invested in the **stock** S by time t.

Definition (Trading strategy)

The couple η is said to be a **trading strategy** if it is $\mathcal{F}$-adapted and

$$\mathbb{E}\left[\int_0^T |\eta^0_t r_t B_t| dt\right] < \infty,$$

$$\mathbb{E}\left[\int_0^T |\eta^1_t \mu(t, S_t) S_t| dt\right] < \infty,$$

$$\mathbb{E}\left[\int_0^T |\eta^1_t \sigma(t, S_t) S_t|^2 dt\right] < \infty.$$

Following a strategy η will result in some wealth described by the **value** or **wealth process**.

Definition (Value of the portfolio)

The **value** of a portfolio with strategy η is given by

$$V_t^{\eta} = \eta_t^0 B_t + \eta_t^1 S_t, \quad t \in [0, T].$$

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Definition (Self-financing strategy)

We say that η is a **self-financing strategy** if

$$dV_t^{\eta} = \eta_t^0 dB_t + \eta_t^1 dS_t, \quad t \in [0, T], \tag{7}$$

meaning that changes in the value of V are caused by gains from the trade, that is changes in the values of the assets.

At this point, having a model for B and S, we can easily express the (semimartingale) dynamics of V using (6) and (5). Indeed, V satisfies the following SDE

$$dV_t^\eta = \left(\eta_t^0 r_t B_t + \eta_t^1 \mu(t, S_t) S_t\right) dt + \eta_t^1 \sigma(t, S_t) S_t dW_t.$$

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Example

$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$, $B_t = e^{rt}$, $\eta_t^0 = 0$ and $\eta_t^1 = 1$. Then

$$V_t^\eta = V_0^\eta + \int_0^t \eta_s^1 dS_s = S_t.$$

Or if $\eta_t^0 = -0.5$ and $\eta_t^1 = 2$ then $V_t^\eta = V_0^\eta - 0.5 B_t + 2 S_t$.

Example

Take $r = 0.03$, $\mu = 0.05$, $\sigma = 0.1$, $S_0 = 100$ €, $V_0^\eta = 50$€, $\eta_t^0 = 0$ and $\eta_t^1 = \mathbb{I}\{S_t < 50\}$.

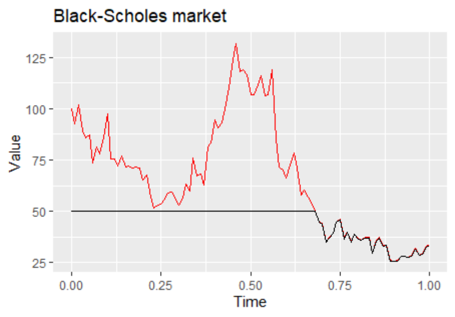
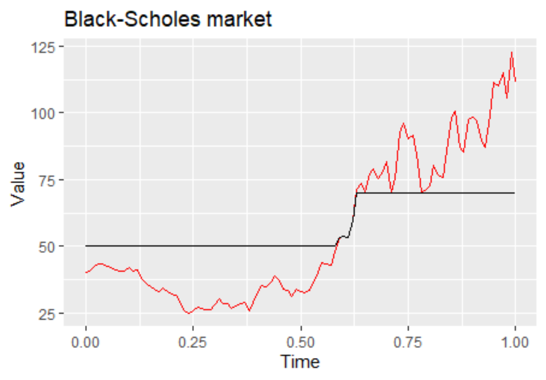


Figure: A market outcome with the corresponding value of the portfolio.

Example

$S_0 = 40$ and $\eta^1_t = \mathbb{I}\{50 < S_t < 70\}$. We buy above 50 and sell at 70.



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Discounting

Discounting

is the relative value of an asset compared to another. Since $B_t = e^{rt}$ is a **risk-less** asset, it makes sense to discount w.r.t. it.
Since B is $\mathbb{P}$-a.s. strictly positive we can define the discounted price process as

$$\widetilde{S}_t = \frac{S_t}{B_t}, \quad t \in [0, T].$$

Obviously $\widetilde{B}_t = 1$. Finally, the **discounted value** of the portfolio associated to a strategy η is given by

$$\widetilde{V}_t^\eta = \frac{V_t^\eta}{B_t}, \quad t \in [0, T].$$

Fundamental theorems of asset pricing

Definition (Arbitrage opportunity)

An **arbitrage opportunity** is a self-financing strategy η with

$$V_0^\eta = 0, \quad V_T^\eta \geq 0, \quad \mathbb{P}[V_T^\eta > 0] > 0.$$

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Remark

The above notion of arbitrage is rather narrow. There is a broader notion known with the name **no free lunch with vanishing risk (NFLVR)**, the mathematical formulation of which is rather technical. In easier terms, a free lunch with vanishing risk is when a sequence of self-financing portfolios which converge to an arbitrage strategy and allows the approximation of a self-financing portfolio (the free lunch and with no risk). NFLVR is the no-arbitrage argument against that possibility.

Absence of arbitrage (in the broader sense of NFLVR) can be rephrased in purely mathematical terms. The corresponding theorem interconnects the absence of arbitrage with the concept of martingale.

Theorem (First Fundamental Theorem of Asset Pricing)

The following statements are equivalent:

(i) *There are no arbitrage opportunities.*

(ii) *There exists an equivalent martingale measure (EMM), i.e. some probability measure* $\mathbb{Q} \sim \mathbb{P}$ *such that the discounted price process* $\widetilde{S}$ *is a* $(\mathbb{Q}, \mathcal{F})$*-martingale.*

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We will price claims using the above fundamental theorem. For that we will need to find such EMM $\mathbb{Q}$ and therefore the following important result.

Theorem (Cameron-Martin-Girsanov theorem)

Let $W = \{W_t, t \in [0,T]\}$ *be a Wiener process on* $(\Omega, \mathcal{F}_T, \mathbb{P})$. *Let* $X = \{X_t, t \in [0,T]\}$ *be an* $\mathcal{F}$*-adapted process with* $X_0 = 0$.
Define the Doléans-Dade exponential $\mathcal{E}(X)$ *of* X *with respect to* W,

$$\mathcal{E}(X)_t = \exp\left(X_t - \frac{1}{2}[X]_t\right), \quad t \in [0,T],$$

where $[X]$ *is the quadratic variation of* X. *If* $\mathcal{E}(X)$ *is a strictly positive martingale, a probability measure* $\mathbb{Q}$ *can be defined on* $(\Omega, \mathcal{F}_T)$ *given by the Radon-Nikodym derivative*

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \mathcal{E}(X)_t, \quad t \in [0,T].$$

Theorem (Cameron-Martin-Girsanov theorem)

Then for each $t \in [0, T]$ the measure $\mathbb{Q}$ restricted to $\mathcal{F}_t$ is equivalent to the measure $\mathbb{P}$ restricted to $\mathcal{F}_t$. Furthermore, if Y is a $(\mathbb{P}, \mathcal{F})$-local martingale, then the process

$$\widetilde{Y}_t \triangleq Y_t - [Y, X]_t, \quad t \in [0, T],$$

is a $(\mathbb{Q}, \mathcal{F})$-local martingale.

Remark

Girsanov's theorem is important in the general theory of stochastic processes since it enables the key result that if $\mathbb{Q}$ is an **absolutely continuous measure** with respect to $\mathbb{P}$ then every $\mathbb{P}$**-semimartingale** is a $\mathbb{Q}$**-semimartingale.**

Corollary

If X in the previous theorem is given by $X_t = \int_0^t u_s dW_s$, $t \in [0,T]$, then $[X,W]_t = \int_0^t u_s ds$, $t \in [0,T]$, and we have that

$$\mathcal{E}(X)_t = \exp\left(\int_0^t u_s dW_s - \frac{1}{2}\int_0^t u_s^2 ds\right), \quad t \in [0,T],$$

and

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} = \mathcal{E}(X)_t, \quad t \in [0,T].$$

Furthermore, if W is a $(\mathbb{P},\mathcal{F})$-Brownian motion, then the process

$$\widetilde{W}_t \triangleq W_t - [X,W]_t = W_t - \int_0^t u_s ds, \quad t \in [0,T],$$

is a $(\mathbb{Q},\mathcal{F})$-Brownian motion.

Proof.

We have $\widetilde{W}_0 = W_0 = 0$. By Girsanov theorem it is a $(\mathbb{Q}, \mathcal{F})$-local martingale and its quadratic variation is

$$[\widetilde{W}, \widetilde{W}]_t = [W, W]_t = t.$$

Then it follows by Lévy's characterization of Brownian motion that this is a $\mathbb{Q}$-Brownian motion. ■

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$$\widetilde{S}_t = \mathbb{E}_{\mathbb{Q}}[\widetilde{S}_T | \mathcal{F}_t].$$

We will see that for the class of trading strategies η, the discounted value of a portfolio, i.e. $\widetilde{V}^{\eta}$, is also a $\mathbb{Q}$-martingale, i.e.

$$\widetilde{V}_t^{\eta} = \mathbb{E}_{\mathbb{Q}}[\widetilde{V}_T^{\eta} | \mathcal{F}_t].$$

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But now if we choose a trading strategy η (if it exists) such that $\widetilde{V}_T^{\eta} = \widetilde{H} = \frac{H}{B_T}$. Then we have

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We say that a claim H is **attainable** if there exists a trading strategy η such that

$$V_T^{\eta} = H, \quad \mathbb{P}-\text{a.s.}$$

where H is an $\mathcal{F}_T$-measurable random variable with $\mathbb{E}[H^2] < \infty$. In such case we say that η is a **replicating strategy** or **perfect hedge** for H.

Definition (Completeness)

We say that the market is **complete** if all contingent claims in the sense explained above are **attainable**.

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Theorem (Second Fundamental Theorem of Asset Pricing)

An arbitrage-free market is **complete** *if, and only if there exists a* **unique EMM**.

Usually, in a more informal setting, **completeness** is related to the **number of risky assets** you consider in the market (or noises driving the assets) and the **number of the assets** that you can actually **trade.** The market we are considering has only **one risky asset** S and it can be **traded**, hence this market is **complete.**

Usually, in a more informal setting, **completeness** is related to the **number of risky assets** you consider in the market (or noises driving the assets) and the **number of the assets** that you can actually **trade.** The market we are considering has only **one risky asset** S and it can be **traded**, hence this market is **complete.**

Let us then try to find such **EMM** $\mathbb{Q}$! (also called **risk-neutral measure**). To this end, we will derive the dynamics of the discounted price process S and value of the portfolio V^{η}.

For computational simplicity, we will consider **investment fractions** rather than units η. Recall that η^0 was the number of units invested in B and η^1 in S. Instead, we now consider the **proportions** invested in B, denoted by π^0 and the **proportion** invested in S, denoted by π^1. They are defined as

$$\pi_t^0 \triangleq \frac{\eta_t^0 B_t}{V_t^\eta}, \quad \pi_t^1 \triangleq \frac{\eta_t^1 S_t}{V_t^\eta}, \quad t \in [0, T],$$

and obviously

$$\pi_t^0 + \pi_t^1 = 1, \quad \mathbb{P} - \text{a.s.}$$

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Because of the above fact, we then just consider π_t^1 and forget about π_t^0 which can be recovered as $\pi_t^0 = 1 - \pi_t^1$.

Theorem (Discounted dynamics)

The discounted dynamics of S and V^{η} for a self-financing trading strategy η are given by

$$d\widetilde{S}_t = \widetilde{S}_t \sigma(t, S_t) \left[\theta_t dt + dW_t\right],$$
$$d\widetilde{V}_t^{\eta} = \widetilde{V}_t^{\eta} \sigma(t, S_t) \pi_t^1 \left[\theta_t dt + dW_t\right],$$

where

$$\theta_t \triangleq \frac{\mu(t, S_t) - r_t}{\sigma(t, S_t)}$$

is called the **market price of risk** *process.*

Proof.

Proof on the blackboard. ∎

The above theorem suggests that $\widetilde{S}$ and $\widetilde{V}^\eta$ are **local martingales** under a measure $\mathbb{Q}$ under which $\widetilde{W}_t \triangleq \int_0^t \theta_s ds + W_t$ is a $(\mathbb{Q}, \mathcal{F})$-Brownian motion. Choosing $u_t = -\theta_t$ in Girsanov's theorem we have

$$\mathcal{E}(X)_t = \exp\left(-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right), \quad t \in [0, T],$$

and $\mathbb{Q}$ is then defined as

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \mathcal{E}(X)_t, \quad t \in [0, T]. \tag{8}$$

Remark (Change of measure requires martingality of $\{\mathcal{E}(X)_t\}_{t\in[0,T]}$)

In order to be able to apply Girsanov's theorem and obtain a true probability measure in (8), we need that $\{\mathcal{E}(X)_t\}_{t\in[0,T}$ is a **martingale.** To check for this is usually very difficult, if not impossible. There are plenty of criteria and conditions for u so that $\mathcal{E}(\int_0^\cdot u_s dW_s)$ is a martingale. The easiest **sufficient condition** for u is the so-called Novikov's condition:

$$\mathbb{E}\left[e^{\frac{1}{2}\int_0^T |u_s|^2 ds}\right] < \infty \quad \text{(Novikov condition)}.$$

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Nevertheless, this condition is **far from being necessary** and it is rather strong. We will this assume that our process θ is such that $\{\mathcal{E}(X)_t\}_{t\in[0,T}$ from (8) is a **martingale.**

Then it follows that

$$\widetilde{W}_t = W_t + \int_0^t \theta_s ds, \quad t \in [0, T],$$

is a $(\mathbb{Q}, \mathcal{F})$-Brownian motion.
Then the $\mathbb{Q}$-dynamics of $\widetilde{S}$ and $\widetilde{V}^\eta$ are

$$\begin{aligned} d\widetilde{S}_t &= \widetilde{S}_t \sigma(t, S_t) d\widetilde{W}_t, \\ d\widetilde{V}_t^\eta &= \widetilde{V}_t^\eta \sigma(t, S_t) \pi_t^1 d\widetilde{W}_t. \end{aligned}$$

We see that they are $\mathbb{Q}$-local martingales since they are the stochastic integral with respect to the martingale $\widetilde{W}$. They are $\mathbb{Q}$-martingales if

$$\mathbb{E}_{\mathbb{Q}}\left[\int_0^T |\widetilde{S}_t \sigma(t, S_t)|^2 dt\right] < \infty, \quad \mathbb{E}_{\mathbb{Q}}\left[\int_0^T |\widetilde{V}_t^\eta \sigma(t, S_t) \pi_t^1|^2 dt\right] = \mathbb{E}_{\mathbb{Q}}\left[\int_0^T |\widetilde{S}_t \sigma(t, S_t)$$

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&\mathbb{E}_{\mathbb{Q}}\left[\int_0^T |\widetilde{V}_t^{\eta} \sigma(t, S_t) \pi_t^1|^2 dt\right] = \mathbb{E}_{\mathbb{Q}}\left[\int_0^T |\widetilde{S}_t \sigma(t, S_t) \eta_t^1|^2 dt\right] < \infty.
\end{aligned}
$$

Hence, $\mathbb{Q}$ defined as in (8) is an **equivalent martingale measure** under which $\widetilde{S}$ is a martingale. Moreover, for a self-financing trading strategy η, $\widetilde{V}^{\eta}$ is also a $\mathbb{Q}$-martingale and hence,

$$\widetilde{V}_t^{\eta} = \mathbb{E}_{\mathbb{Q}}[\widetilde{V}_T^{\eta}|\mathcal{F}_t], \quad t \in [0, T].$$

If H is a **contingent claim** and η a **self-financing replicating portfolio** for H then

$$\widetilde{V}_t^{\eta} = \mathbb{E}_{\mathbb{Q}}[\widetilde{H}|\mathcal{F}_t], \quad t \in [0, T]. \tag{9}$$

In particular, the initial **arbitrage-free price** of H is given by

$$\widetilde{V}_0^{\eta} = \mathbb{E}_{\mathbb{Q}}[\widetilde{H}].$$

Exercise (Black-Scholes formula of call option)

Assume the following market dynamics

$$dB_t = rB_t dt, \quad B_0 = 1, \quad t \in [0, T],$$

and

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \quad S_0 > 0, \quad t \in [0, T].$$

Consider the **option** *that pays* $H = (S_T - K)_+$ *where* $(x)_+ = \max\{x, 0\}$. *Find the measure* $\mathbb{Q}$ *under which* $\widetilde{S}$ *is a* **martingale** *and the* **price** *of* H *at time* t.

Deriving a PDE for the hedging portfolio

We restrict ourselves to European options, i.e. $H(S_T)$ for a function $H : \mathbb{R} \to \mathbb{R}$ such that $\mathbb{E}[H(S_T)^2] < \infty$. Let us recover relation (9),

$$\widetilde{V}_t^\eta = \mathbb{E}_{\mathbb{Q}}[\widetilde{H}|\mathcal{F}_t], \quad t \in [0, T].$$

The process $\widetilde{V}_t^\eta$, $t \in [0, T]$ is by definition a $\mathbb{Q}$-martingale. In terms of V_t^η and $H(S_T)$ we have

$$\frac{V_t^\eta}{B_t} = \mathbb{E}_{\mathbb{Q}}\left[\frac{H(S_T)}{B_T}\middle|\mathcal{F}_t\right], \quad t \in [0, T]. \tag{10}$$

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Using that S is a Markov process we have,

$$V_t^\eta = B_t \mathbb{E}_{\mathbb{Q}}\left[\frac{H(S_T)}{B_T}\Big|\mathcal{F}_t\right] = B_t \mathbb{E}_{\mathbb{Q}}\left[\frac{H(S_T)}{B_T}\Big|\sigma(S_t)\right]$$

and by definition of conditional expectation we have that

$$V_t^\eta = B_t F(t, S_t)$$

for some measurable function F. So V is itself a function of t and S_t.

From now on, let us denote it by $V(t, S_t)$. So far,

$$V(t, S_t) = B_t F(t, S_t). \tag{11}$$

Applying **Itô's formula** to F we have

$$dF(t, S_t) = \left[\partial_t F(t, S_t) + S_t \mu(t, S_t) \partial_x F(t, S_t) + \frac{1}{2} S_t^2 \sigma(t, S_t)^2 \partial_x^2 F(t, S_t) \right] dt \\ + S_t \sigma(t, S_t) \partial_x F(t, S_t) dW_t. \tag{12}$$

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$$+ S_t \sigma(t, S_t) \partial_x F(t, S_t) dW_t. \tag{12}$$

We can recast equation (12) under $\mathbb{Q}$ by using $dW_t = d\widetilde{W}_t - \theta_t dt$. Hence,

$$dF(t, S_t) = \left[\partial_t F(t, S_t) + S_t \mu(t, S_t) \partial_x F(t, S_t) + \frac{1}{2} S_t^2 \sigma(t, S_t)^2 \partial_x^2 F(t, S_t)\right.$$
$$\left. - S_t \sigma(t, S_t) \theta_t \partial_x F(t, S_t)\right] dt + S_t \sigma(t, S_t) \partial_x F(t, S_t) d\widetilde{W}_t.$$

But observe that $\sigma(t,S_t)\theta_t = \mu(t,S_t) - r_t$. Hence,

$$dF(t,S_t) = \left[\partial_t F(t,S_t) + S_t r_t \partial_x F(t,S_t) + \frac{1}{2}S_t^2\sigma(t,S_t)^2\partial_x^2 F(t,S_t)\right] dt$$
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We know that F is a $\mathbb{Q}$-martingale, so the drift term above must be 0. That is

$$\partial_t F(t, S_t) + S_t r_t \partial_x F(t, S_t) + \frac{1}{2} S_t^2 \sigma(t, S_t)^2 \partial_x^2 F(t, S_t) = 0. \tag{14}$$

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$$\partial_t V(t, x) + r_t x \partial_x V(t, x) + \frac{1}{2} x^2 \sigma(t, x)^2 \partial_x^2 V(t, x) = r_t V(t, x), \tag{15}$$

with boundary condition $V(T, x) = H(x)$.

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$$\partial_t V(t,x) + r_t x \partial_x V(t,x) + \frac{1}{2} x^2 \sigma(t,x)^2 \partial_x^2 V(t,x) = r_t V(t,x), \tag{15}$$

with boundary condition $V(T,x) = H(x)$.
This **PDE** is known as **Black-Scholes PDE** for pricing the **European option** $H(S_T)$. The formula that relates conditional expectations of functionals of Markov processes with solutions of second order PDE's is known as the **Feynman-Kac formula**.

Exercise (Pricing Asian options)

Derive the corresponding PDE for the price of an **Asian option**. *Hint: An Asian option is an option with payoff being a function of* $\frac{1}{T}\int_0^T S_u du$, *i.e.* $H = H\left(\frac{1}{T}\int_0^T S_u du\right)$. *Then use that* $\int_0^T S_u du = \int_0^t S_u du + \int_t^T S_u du$, *the fact that the first term is* $\mathcal{F}_t$*-measurable, the Markov property of* S *and the bivariate Itô's formula.*

Exercise (Real world pricing)

In the theory we have used, we construct a measure $\mathbb{Q}$ (risk-neutral measure or EMM) under which discounted prices are a $\mathbb{Q}$-martingale. This allows to find a "martingale" formula which provides the fair price. In this exercise, instead of finding $\mathbb{Q}$ such that $\frac{S_t}{B_t}$, $t \in [0, T]$ is a $\mathbb{Q}$-martingale, find a reference **numéraire** *(which is not B anymore) V^{η^*} for a right trading strategy η^* such that $\frac{S}{V^{\eta^*}}$, $t \in [0, T]$ is a $\mathbb{P}$-martingale and provide a* **real world pricing formula***, that is a pricing formula under $\mathbb{P}$ and not under $\mathbb{Q}$.*

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Mathematical Finance

STK4500 Life insurance and finance