UiO : Department of Mathematics
University of Oslo

STK4500: Life insurance and finance

Mortality basis Norway

David R. Banos

Spring 2024

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Norwegian mortality 1846-2022

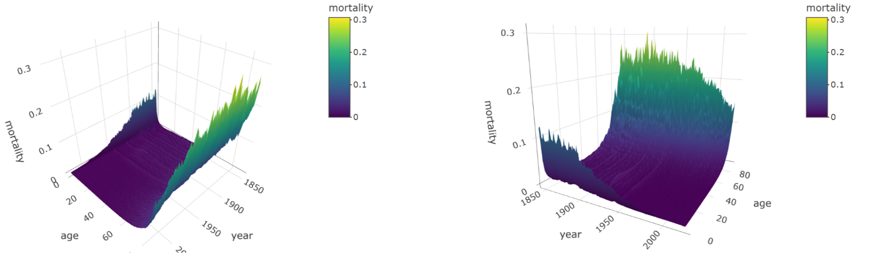

Figure: Historical mortality from 1846 to 2022 for ages ranging from 0 to 90.

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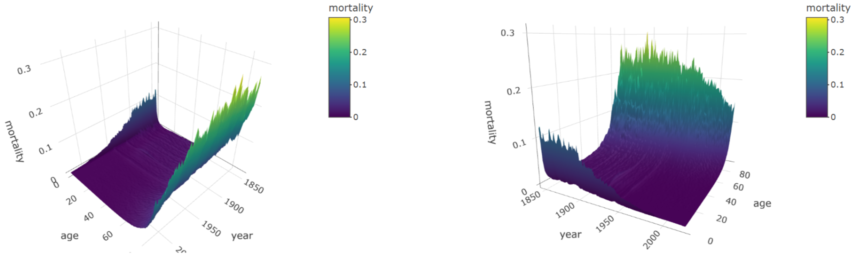

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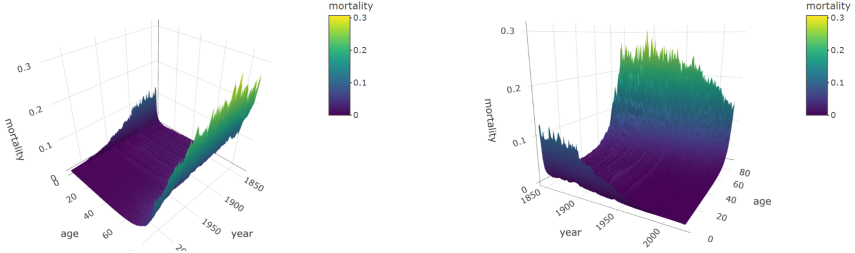

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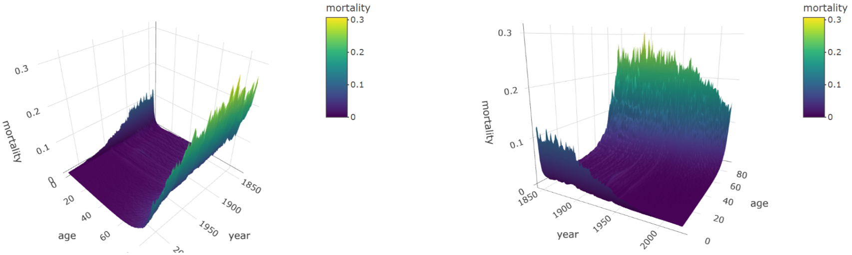

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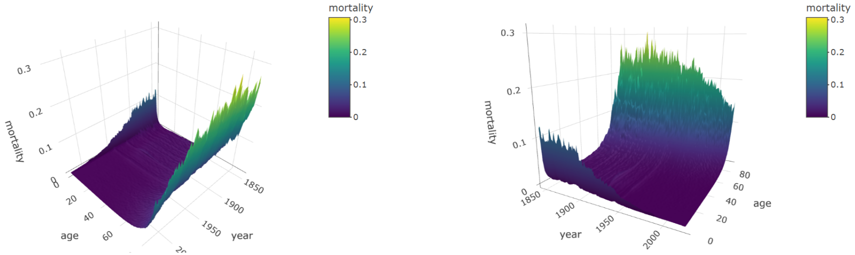

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- Mortality increases with age.
- Mortality decreases with calendar years.
- Infant mortality has decreased drastically with calendar years.
- We can appreciate some "bumps", for example the 1918-1920 flu pandemic (Spanish flu) and 2WW.

Classical mortality model: Lee-Carter

$$\mu(x,t) = \{\text{mortality at age } x \text{ in calendar year } t\}.$$

The Lee-Carter model assumes that

$$\log(\mu(x,t)) = a_x + k_t b_x + \varepsilon(x,t),$$

where

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3. k_t and b_x are found using the principal components of the matrix $\log(\mu(x,t)) - a_x$, since the matrix $\mu(t,x)$ has highly correlated entries.
4. $\varepsilon(x,t)$ are the residuals. The above method minimizes the squares.

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Equivalent expressions:

$$p_{**}(x, x+s) = \exp\left\{-\int_x^{x+s} \mu(u, Y-x+u)du\right\},$$

or

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where Y_0 is the birth year.

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Finanstilsynet provides a standard model for $\mu(x, Y)$.

To sum up,

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- Use numerical integration for the integral.
- We go now to https://www.finanstilsynet.no/nyhetsarkiv/pressemeldinger/2013/nytt-dodelighetsgrunnlag-i-kollektiv-pensjonsforsikring/

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where t is a calendar year larger than 2013. In the formula above, $w(x)$ is a weight which is given by

$$w(x) = \min\left\{2.671548 - 0.172480x + 0.001485x^2, 0\right\}, \quad \text{for males}$$
$$w(x) = \min\left\{1.287968 - 0.101090x + 0.000814x^2, 0\right\}, \quad \text{for females.}$$

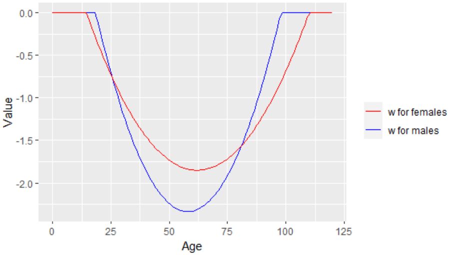

Figure: The function w for females (in blue) and males (in red).

$$\mu_{Kol}(x,t) = \mu_{Kol}(x,2013)\left(1 + \frac{w(x)}{100}\right)^{t-2013}, \tag{2}$$

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- The higher t gets, the lower $\mu_{Kol}(x,t)$ gets.
- This is only applied to working and pension ages: 16 to 100 for males and 110 for females.

Two types of risk: longevity and mortality.

	Longevity risk
males	$1000\mu_{Kol}(x, 2013) = 0.189948 + 0.003564 \cdot 10^{0.051x}$
females	$1000\mu_{Kol}(x, 2013) = 0.067109 + 0.002446 \cdot 10^{0.051x}$

Table: Mortality basis from K2013 for longevity risk

	Mortality risk
males	$1000\mu_{Kol}(x, 2013) = 0.241752 + 0.004536 \cdot 10^{0.051x}$
females	$1000\mu_{Kol}(x, 2013) = 0.085411 + 0.003114 \cdot 10^{0.051x}$

Table: Mortality basis from K2013 for mortality risk

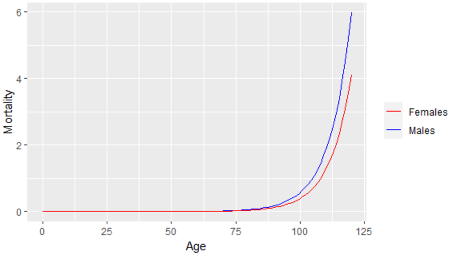
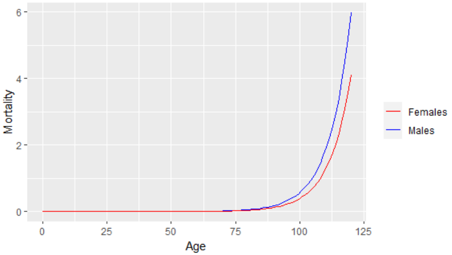


Figure: Mortalities $\mu_{Kol}(x, 2013)$ for both legal genders for calendar year 2013. We observe a lower mortality for females.



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Remark

The insurer will usually carry out solvency computations for the two types of risk: longevity and mortality, then look at the differences.

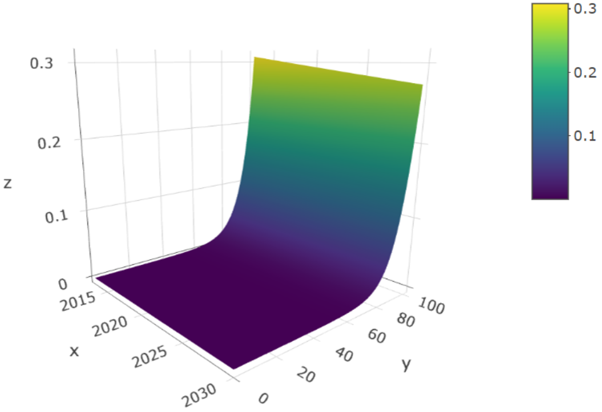

Figure: Surface from K2013 for ages 0 to 100 and years 2013 to 2030 for females.

Year 2024

Choose $t = 2024$ and then use

$$\mu_{Kol}(x, 2024) = \mu_{Kol}(x, 2013)\left(1 + \frac{w(x)}{100}\right)^{11}$$

and for $\mu_{Kol}(x, 2013)$ we would choose the right one by specifying a risk type and a legal gender from the previous slides.

Year 2024

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Remark

Remember that $t = 2024$ needs to be updated every year!

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R files: *K2013.R* *probabilityK2013.R*

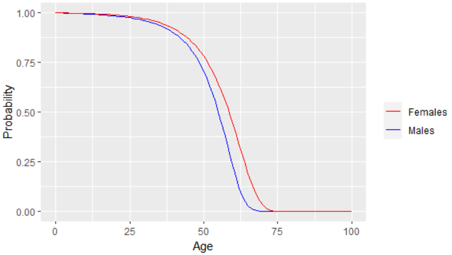

Figure: Survival probabilities for a 30 year old Norwegian tax payer in year $Y = 2024$ for both legal genders.

$$\text{Function: } t \mapsto p_{**}^{30}(0,t) = \exp\left\{\int_0^t \mu_{Kol}(30+u, 2024+u)du\right\}.$$

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3. Therefore, the distribution of T_z is

$$F_{T_z}(t) = \mathbb{P}[T_z \leq t] = 1 - \mathbb{P}[T_z > t] = 1 - \exp\left\{\int_0^t \mu_{Kol}(z+u, Y+u)du\right\}.$$

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4. We know that $F_{T_z}(T_z) = U$, U uniform on $[0, 1]$. Hence $T_z = F^{-1}_{T_z}(U)$.
5. Hence, a random life $t^* \sim T_z$ can be obtained via

$$u^* \sim U \Rightarrow F^{-1}_{T_z}(u^*).$$

Theorem (Generating a random life time)

Let z be the age of an individual in year Y and T_z denote the remaining life time of this individual. Then, we can generate a random value from T_z using the following algorithm:

1. *Generate a random value u^* from a uniform distribution on $[0, 1]$.*

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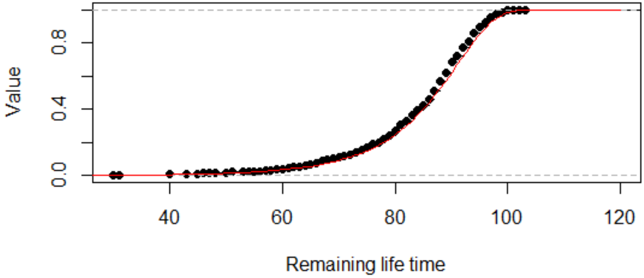

Figure: Comparison of the theoretical distribution of T_z, $z = 30$ in $Y = 2024$ with $R = 0$ (mortality) and $G = 1$ (female). Seed number 1 was used.

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