UiO : **Department of Mathematics**
University of Oslo

STK4500: Life insurance and finance

Riemann-Stieltjes integration (quick guide)

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Riemann integral

Recall:

$$\int_a^b f(s)ds$$

defined by

$$\lim_{n\to\infty} \sum_{t_i\in\pi_n} f(t_i)(t_i - t_{i-1}),$$

where $\{\pi_n\}_{n=1}^{\infty}$ is a sequence of partitions with $a = t_0 < \cdots < t_n = b$ whose mesh $\max_{i=1,\dots,n} |t_i - t_{i-1}| \to 0$.

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Example: if π_n are given by $t_i = \frac{b-a}{n} i$, $i = 0, \dots, n$ then

$$\int_a^b f(s)ds = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f\left(\frac{b-a}{n} i\right)$$

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Newton–Leibniz theorem:

$$\int_a^b f(s)ds = F(b) - F(a),$$

where F is a primitive of f, i.e. $F' = f$.

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Bounded variation (BV)

A function f is said to be of bounded variation on an interval $[a, b]$ if, and only if

$$V_{[a,b]}(f) \triangleq \sup_{\pi \in \mathcal{P}} \sum_{t_i \in \pi} |f(t_i) - f(t_{i-1})| < \infty,$$

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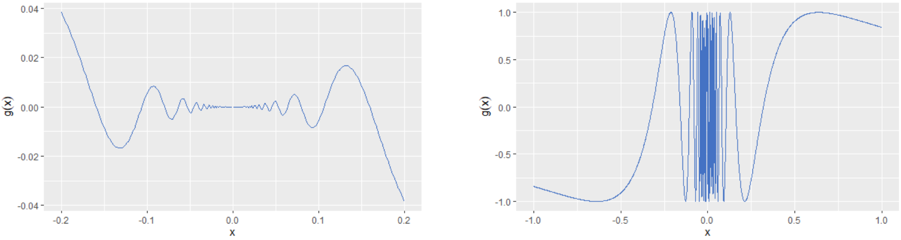

Figure: Example of BV and not BV. Left: $g(x) = x^2 \sin\left(\frac{1}{x}\right)$, $x \neq 0$ and $g(0) = 0$. Right: $g(x) = x \sin\left(\frac{1}{x}\right)$, $x \neq 0$ and $g(0) = 0$

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Extension of Riemann integral:

$$\int_a^b f(s)dg(s)$$

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Sufficient condition for existence: f **continuous** and g of **bounded variation**.

$$\left|\int_{[a,b]} f(s)dg(s)\right| \leq \lim_{n\to\infty} \sum_{t_i\in\pi_n} |f(t_i)(g(t_i) - g(t_{i-1}))| \leq CV_{[a,b]}(g) < \infty,$$

where $C = \max_{s\in[a,b]} |f(s)|$.

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Rules of calculus

We will usually assume that g is a.e. differentiable with a finite number of jumps (discontinuities).

$$\int_{[a,b]} f(s)dg(s) = \int_a^b f(s)g'(s)ds + \sum_{a \leq s \leq b} f(s)\Delta g(s),$$

where $\Delta g(s) = g(s) - g(s-)$ denotes the jumps of g at any time s.

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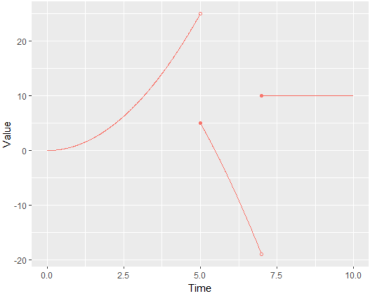

Figure: Example of an a.e. differentiable function with jumps.

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Rules of calculus

$$g(s) = \begin{cases} s^2, & s \in [0,5) \\ 30 - s^2, & s \in [5,7) \\ 10, & s \in [7,\infty). \end{cases}$$

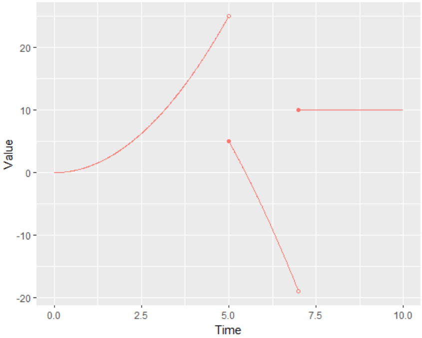

Figure: Graph of the function g.

Rules of calculus

$$g'(s) = \begin{cases} 2s, & s \in [0,5) \\ -2s, & s \in (5,7) \\ 0, & s \in (7,\infty). \end{cases}$$

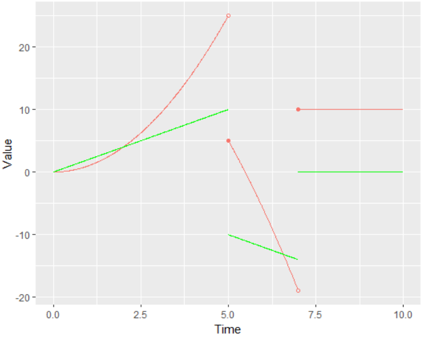

Figure: Function g in red and its derivative g' in green wherever it is differentiable. Jump sizes are: $\Delta g(5) = -20$ and $\Delta g(7) = 29$.

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Hence,

$$\int_{[0,10]} f(s)dg(s) = \int_0^{10} f(s)g'(s)ds + f(5)\Delta g(5) + f(7)\Delta g(7).$$

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For example, if $f(s) = s$,

$$\int_{[0,10]} f(s)dg(s) = \int_0^5 s \cdot 2sds + \int_5^7 s \cdot (-2s)ds + 5 \cdot (-20) + 7 \cdot 29 = 41.$$

Rules of calculus

Summary:

$$\int_{[a,b]} f(s)dg(s) = \int_a^b f(s)g'(s)ds + \sum_{a\leq s\leq b} f(s)\Delta g(s),$$

where $\Delta g(s) = g(s) - g(s-)$ denotes the jumps of g at any time s.

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Consequence: An immediate example is

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if g is constant everywhere except for a jump of size $\Delta g(t)$ at time $t \geq 0$.

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Example: if $g(s) = \mathbb{I}_{[a,\infty)}(s)$, $a \geq 0$, then

$$\int_{[0,\infty)} f(s)dg(s) = f(a).$$

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