UiO : Department of Mathematics
University of Oslo

STK4500: Life insurance and finance

Cash flows and present values

David R. Banos

Spring 2024

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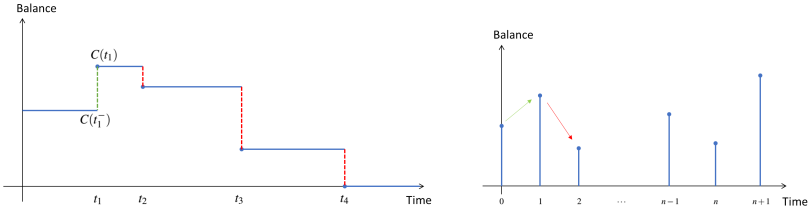
Cash flow



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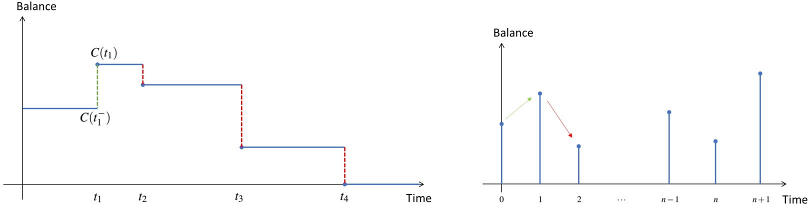

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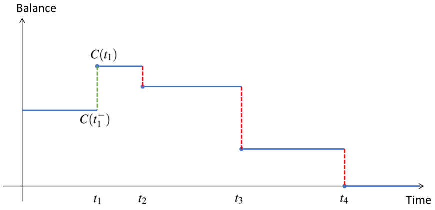

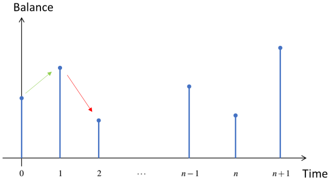

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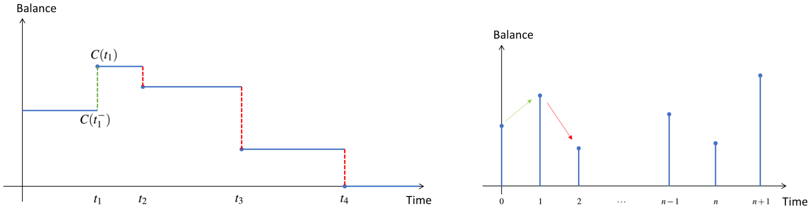

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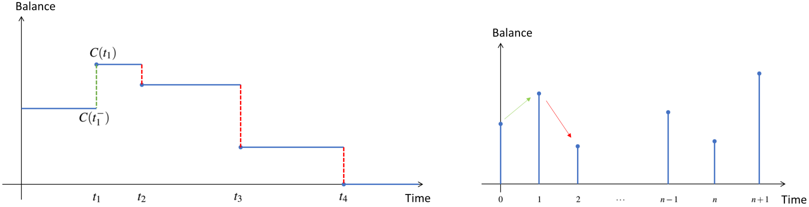

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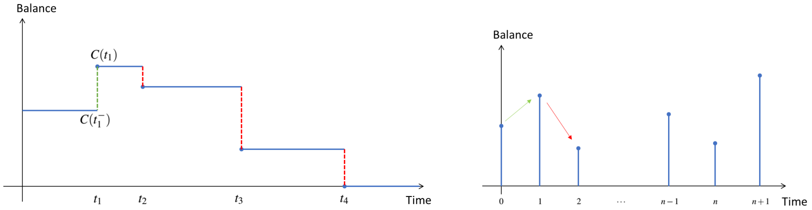

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- We will deal with continuous and discrete time models/formulae.

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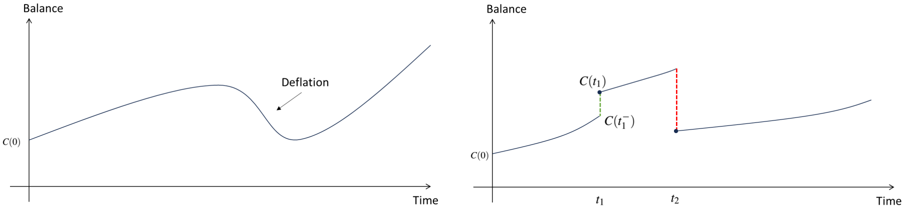

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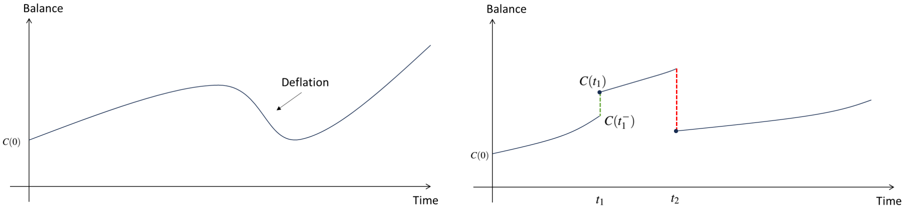

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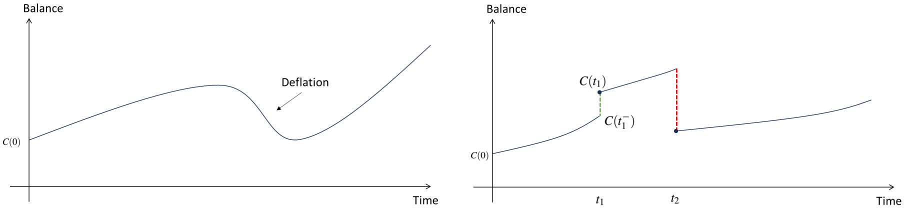

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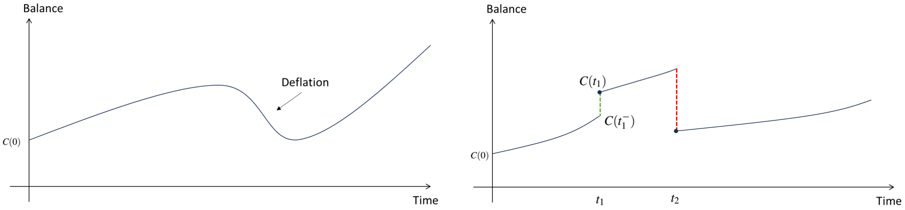

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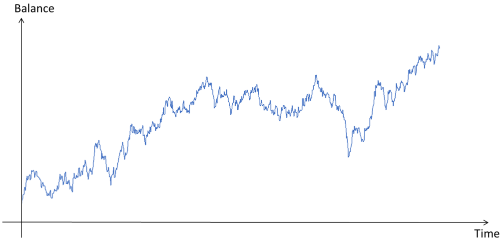
- Inflation $\leftrightarrow$ increase. Deflation $\leftrightarrow$ decrease.

Cash flow

Here is an example of a more irregular cash flow.



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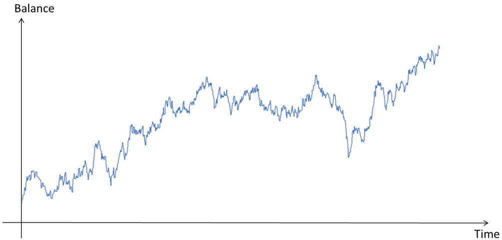

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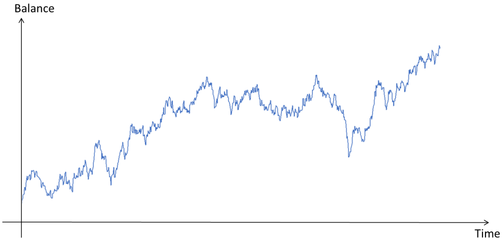

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- One usually uses Brownian motion to model such behaviour.
- Problem: Such graph is not differentiable! Not of bounded variation either.

Definition

Cash flows are simply functions or sequences (continuous vs. discrete).

Definition (Deterministic cash flow)

A **cash flow** C is a function of bounded variation in continuous time, or a sequence of values in discrete time.

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Example

- **Continuous time:** $C(t) = e^{rt}$, $t \geq 0$ (continuous compounding).
- **Discrete time** $C_n = (1 + r)^n$, $n = 0, 1, \dots$ (discrete compounding).
- Or any other function/sequence you may think of.

Definition stochastic cash flow

A **stochastic cash flow** is a cash flow whose outcome is uncertain/random.

Definition (Stochastic cash flow)

A **stochastic cash flow** is a stochastic process whose sample paths are cash flows. That is $C(t, \omega)$, $t \geq 0$, is a continuous time cash flow or $C_n(\omega)$, $n \geq 0$, is a discrete time cash flow.

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Example

- **Continuous time:** $C(t) = e^{rt+Z}$, $t \geq 0$ where Z is normally distributed with mean 0 and variance σ^2.
- **Discrete time:** $C_n = (1 + rZ)^n$, $n = 0, 1, \dots$ where Z is a random variable.
- Or any other stochastic process you may think of.

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Discount factors

Important factor:

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- $v(t)$: today's value of one unit at (continuous) time t.
- $v(n)$: today's value of one unit at (discrete) time n.
- There is always a conversion between

$$(1 + \delta(t))^t = e^{\int_0^t r(s)ds}$$

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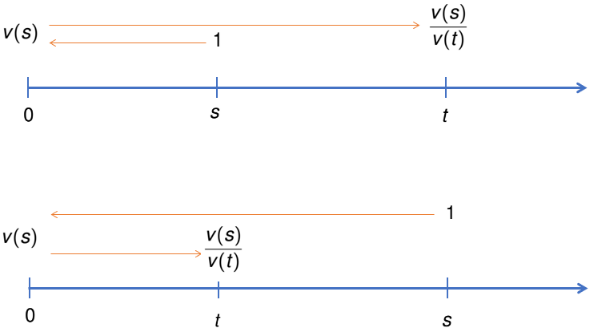
- Saying "L is an asset/liability" gives little information about its true value without knowing when in the timeline it is valued.

Discount factors



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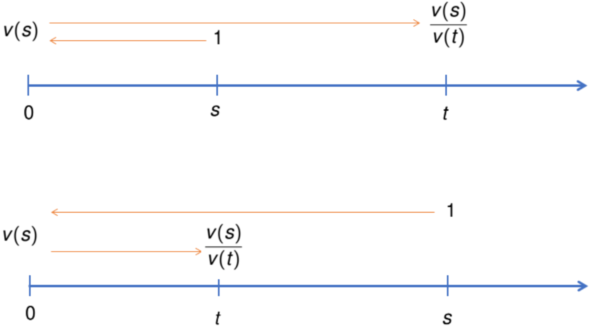

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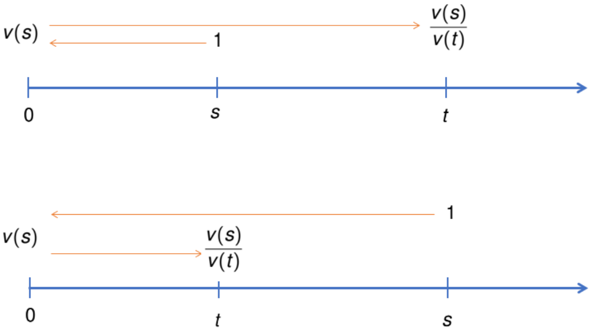

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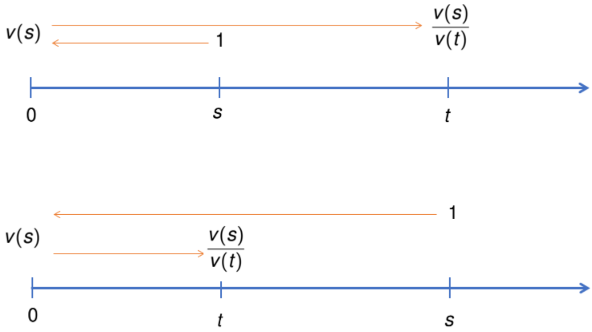

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- If s is running over time, then the total today's value of the whole cash flow C (present value) is

$$\int_{[0,\infty)} v(s)dC(s).$$

PV of cash flows

We can translate the total present value of C to any arbitrary middle time t, i.e. the value of $\int_{[0,\infty)} v(s)dC(s)$ at time t is thus

$$\underbrace{\frac{1}{v(t)}}_{\text{Inflated to time } t} \underbrace{\int_{[0,\infty)} v(s)dC(s)}_{\text{Today's value of } C}.$$

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- Again, it can be translated to any middle time n:

$$\frac{1}{v(n)}\sum_{k=0}^{\infty} v(k)\Delta C_k.$$

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Notations PV, retrospective and prospective value

We introduce the same notations for continuous and discrete time:

- Present value of a cash flow C at time t: $V(t, C)$ or simply $V(t)$.

$$V(t) = \frac{1}{v(t)} \int_{[0,\infty)} v(s) dC(s), \quad V(n) = \frac{1}{v(n)} \sum_{k=0}^{\infty} v(k) \Delta C_k.$$

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- Obvious relation:

$$V(t) = V^-(t) + V^+(t), \quad V(n) = V^-(n) + V^+(n).$$

Example of PV of a cash flow

Let C (continuous time) be given by

$$C(t) = \begin{cases} \text{kr. } 20, & t \in [0,2), \\ \text{kr. } 30, & t \in [2,3), \\ \text{kr. } 5, & t \in [3,7), \\ \text{kr. } 50, & t \in [7,\infty), \end{cases}$$

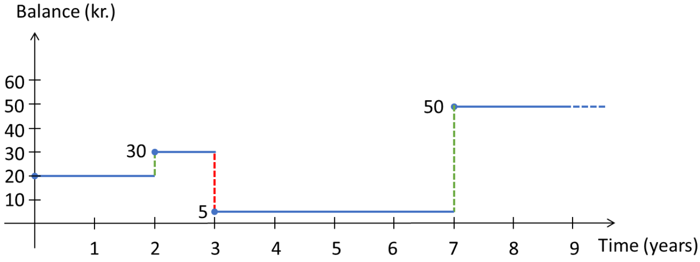

Figure: Example of cash flow (it does not need to be piecewise constant)

Example of PV of a cash flow

Then

$$\Delta C(s) = \begin{cases} \text{kr. } 20, & s = 0, \\ \text{kr. } 10, & s = 2, \\ \text{kr. } -25, & s = 3, \\ \text{kr. } 45, & s = 7, \\ \text{kr. } 0, & \text{otherwise.} \end{cases}$$

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Hence,

$$\begin{aligned} V(0) &= \int_{[0,\infty)} v(s)dC(s) = \sum_{0 \le s < \infty} v(s)\Delta C(s) \\ &= v(0)\Delta C(0) + v(2)\Delta C(2) + v(3)\Delta C(3) + v(7)\Delta C(7) = 43.05 \text{ kr.} \end{aligned}$$

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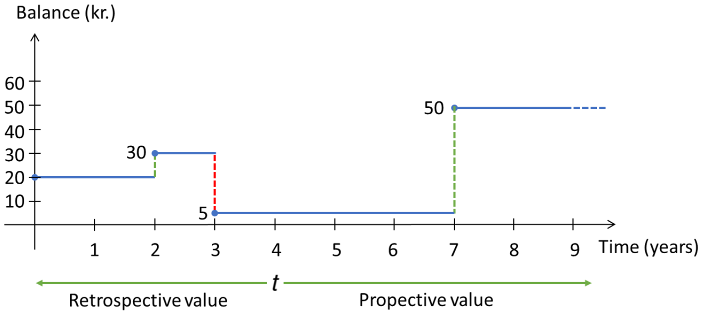
If you promise me this cash flow I should give you 43.05 kr so we are even!

Example of PV of a cash flow

Now look at $t = 4$.



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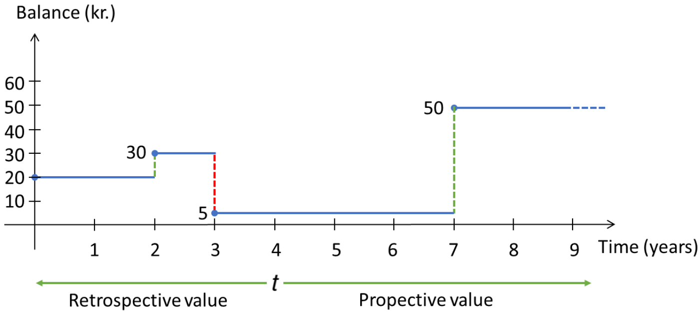

- Idea retrospective: if we stand in $t = 4$, what is the value of what has happened so far at time t?
- Idea prospective: if we stand in $t = 4$, what is the value of the remaining future cash flow that has not taken place yet. In other words, what should you pay me back to cancel the cash flow?

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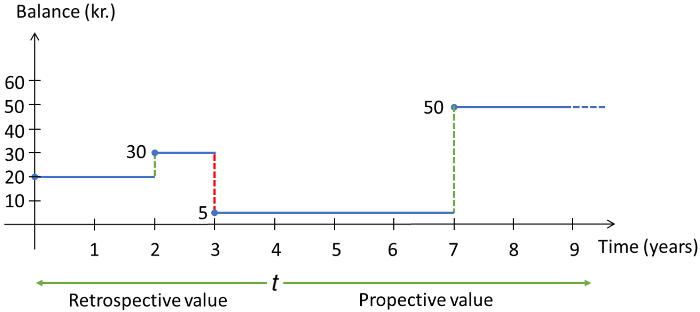

Retrospective:

$$V^{-}(4) = \frac{1}{v(4)} \int_{[0,4]} v(s)dC(s)$$

$$= \frac{1}{v(4)} \left(v(0)\Delta C(0) + v(2)\Delta C(2) + v(3)\Delta C(3) \right) = 7.41 \text{ kr.}$$

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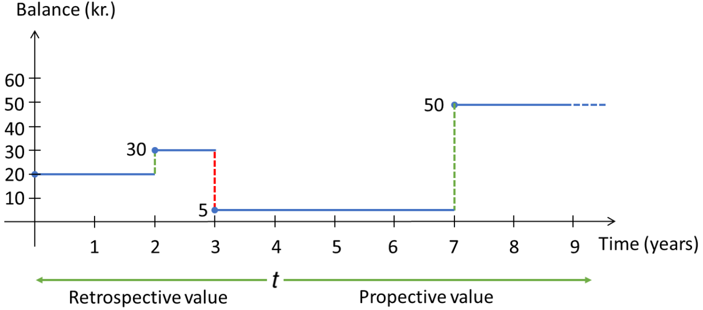

Prospective:

$$V^{+}(4) = \frac{1}{v(4)} \int_{(4,\infty)} v(s) dC(s)$$
$$= \frac{1}{v(4)} v(7) \Delta C(7) = 41.13 \text{ kr.}$$

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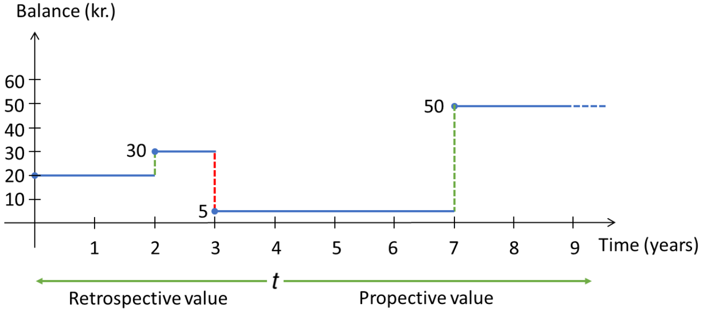
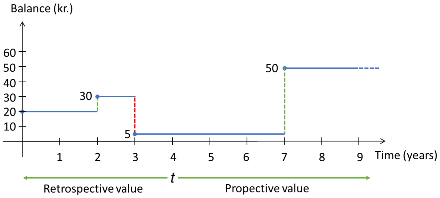


Observation:

$$V(4) = 48.53 \text{ kr.}, \quad V^{-}(4) = 7.41 \text{ kr.}, \quad V^{+}(4) = 41.13 \text{ kr.}$$

$$V(4) = V^{-}(4) + V^{+}(4).$$

Example of PV of a cash flow



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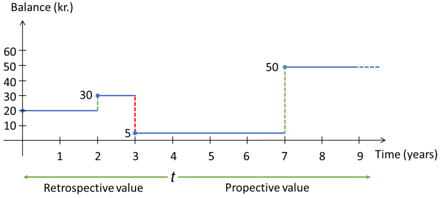

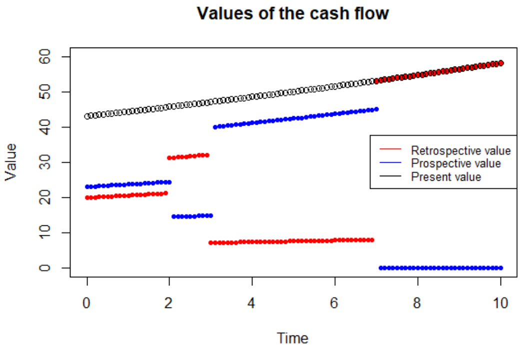

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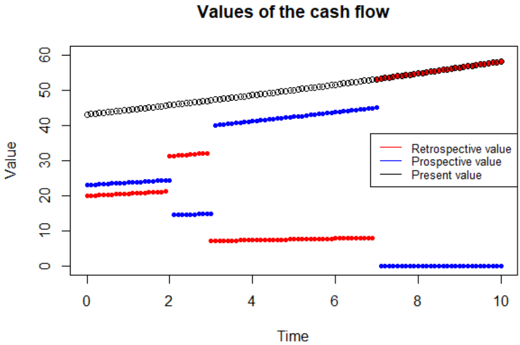

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- Kolmogorov equations: $\frac{d}{ds} P(t, s) = P(t, s)\Lambda(s)$ (fwd.) or $\frac{d}{dt} P(t, s) = -\Lambda(t)P(t, s)$ (bwd.) where P is the transition probability matrix and Λ the transition rate matrix.

Markov setting

Introduce the following stochastic processes:

- In continuous time ($t \geq 0$)

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- We may write I_i and N_{ij} and drop Z when clear.

Policy functions

Definition (Policy functions (discrete time))

Let $a_i, a_{ij} : \mathbb{N} \to \mathbb{R}$, $i, j \in \mathcal{Z}$, be two discrete time functions. We call them **policy functions** whenever they model the following quantities:

- $a_i(n) =$ punctual payments made at time n when the insured is in state i.
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Definition (Policy functions (continuous time))

Let $a_i, a_{ij} : \mathbb{R} \to \mathbb{R}$, $i, j \in \mathcal{Z}$, be two functions of bounded variation. We call them **policy functions** whenever they model the following quantities:

- $a_i(t) =$ the *accumulated* premiums and benefits up to time t while the insured is in state i.
- $a_{ij}(t) =$, $j \neq i$, payments at time t for a switch from state i to state j at time t.

Policy cash flows

Definition (Policy cash flow in discrete time)

Given policy functions a_i, a_{ij}, $i, j \in \mathcal{Z}$, we define the **policy cash flow** at any time $k = 0, 1, \ldots$ by

$$\Delta C_k = \sum_{i \in \mathcal{Z}} I_i^Z(k) a_i(k) + \sum_{i,j \in \mathcal{Z}} \Delta N_{ij}^Z(k) a_{ij}(k).$$

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PV, retrospective and prospective values

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- **Obs:** This is a stochastic cash flow!

PV, retrospective and prospective value (continuous time)

$$V(t,C) = \frac{1}{v(t)} \sum_{i \in \mathcal{Z}} \int_{[0,\infty)} v(s) I_i^Z(s) da_i(s) + \frac{1}{v(t)} \sum_{i,j \in \mathcal{Z}} \int_{[0,\infty)} v(s) a_{ij}(s) dN_{ij}^Z(s),$$

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Example: disability pension with death benefit

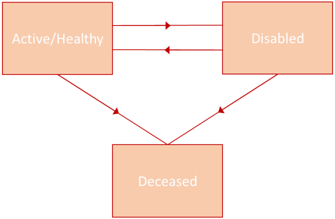

Figure: Markov model of disability pension.

- States $\mathcal{Z} = \{0, 1, 2\}$ where 0 active, 1 disabled and 2 deceased.

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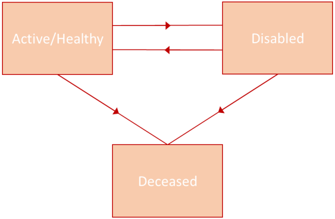

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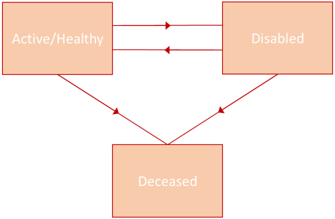

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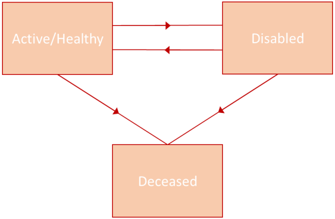

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- Contract duration: $T = 10$ years. Age of insured: $z_0 = 50$ years in 2024.
- **Policy:** Yearly disability pensions of $P = 100\,000$ are paid to the insured while in 1. A final death benefit $B = 1\,000\,000$ is paid to the insured when a transition to 2. Everything stops after T.

Example: disability pension with death benefit

Policy functions:

For sojourns:

$$a_1(t) = \begin{cases} Pt, & t \in [0, T) \\ PT, & t \in [T, \infty) \end{cases} .$$

For transitions:

$$a_{02}(t) = \begin{cases} B, & t \in [0, T] \\ 0, & t \notin [0, T] \end{cases} , \quad a_{12}(t) = \begin{cases} B, & t \in [0, T] \\ 0, & t \notin [0, T] \end{cases} .$$

Example: disability pension with death benefit

Recall:

$$dC(s) = \sum_{i \in \mathcal{Z}} I_i^Z(s)da_i(s) + \sum_{i,j \in \mathcal{Z}} dN_{ij}^Z(s)a_{ij}(s).$$

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Hence, the prospective value is

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- Pick j_n from $\mathcal{Z} = \{0, 1, 2\}$ randomly according to the vector $\{p_{j_{n-1} 0}((n-1)h, T), p_{j_{n-1} 1}((n-1)h, T), p_{j_{n-1} 2}((n-1)h, T)\}$. Set $Z(T) = j_n$.

Example: disability pension with death benefit

We simulated 4 policy holders:

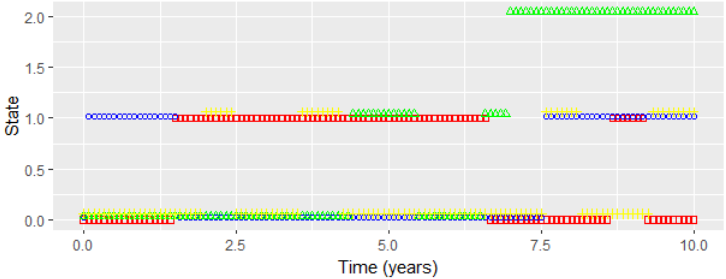

Figure: 4 random outcomes.

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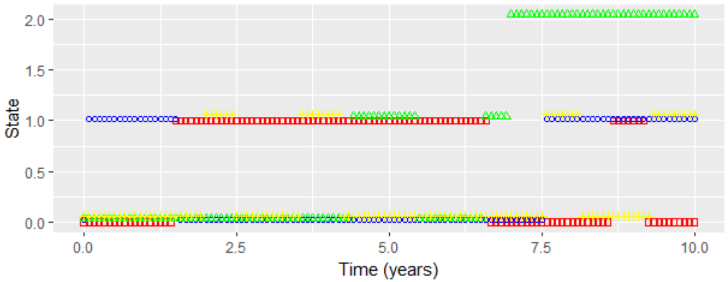
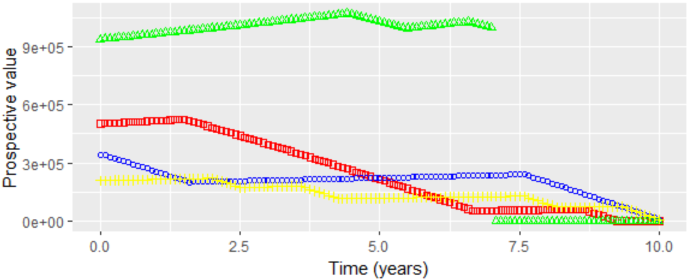


Figure: 4 random outcomes.

We see that most likely, the green outcome has the highest value: some disability pensions twice and a death benefit of 1MNOK around policy year 7.

Example: disability pension with death benefit

Here we plot the function $t \mapsto V^{+}(t, \omega_i)$, $i = 1, 2, 3, 4$.



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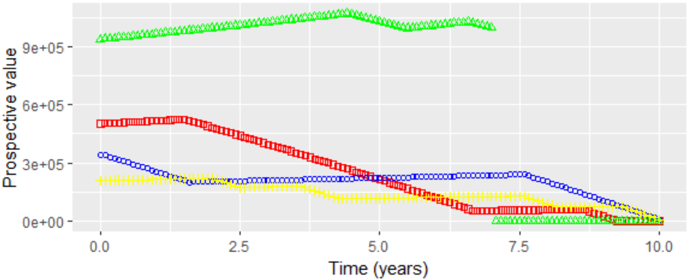

Figure: 4 random prospective values.

We confirm that the cash flow of the «green» policyholder has highest value and that a payment of 1MNOK is made around year 7.

Example: disability pension with death benefit

We see that the green policyholder was the most expensive among these four. If we look at their outcome we see:

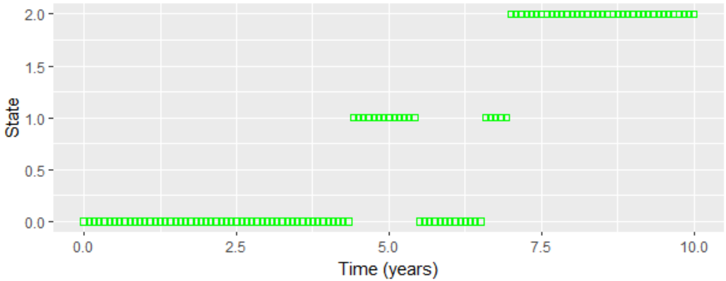

Figure: One of the four outcomes (number 3) who passed away around age 57 and one month.

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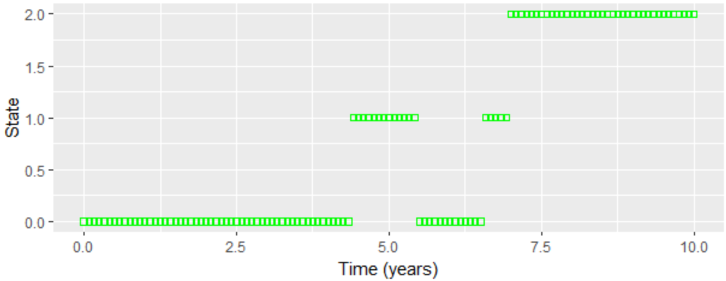

Figure: One of the four outcomes (number 3) who passed away around age 57 and one month.

Actually, policyholder nr. 3 870 was the most expensive (113 months disability and one death benefit), while nr. 276 was the cheapest (always stayed in state 0). We used seed 1.

Example: disability pension with death benefit

Now we look at the mean of all prospective values of 100 random outcomes:

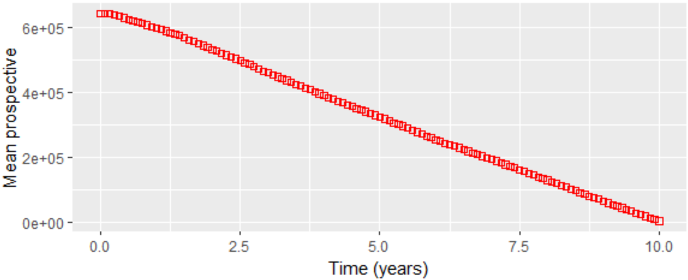

Figure: Mean prospective values of 10 000 generated random insurance cash flows (simulation time: 2.06361 mins). Initial value: 640 940.1 .

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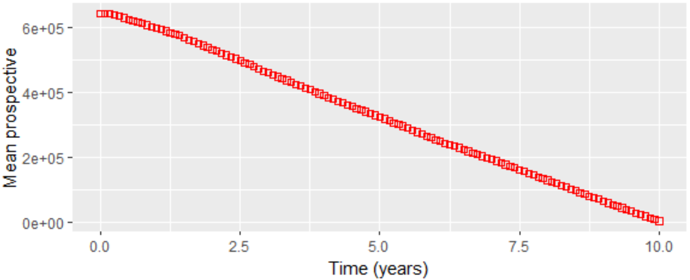

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- We see that as time goes by, the value decreases since we are approaching the end of the contract.

Example: disability pension with death benefit

Out of curiosity: we plot the same mean prospective values based on 10 000 simulations *with* and *without* death benefit.

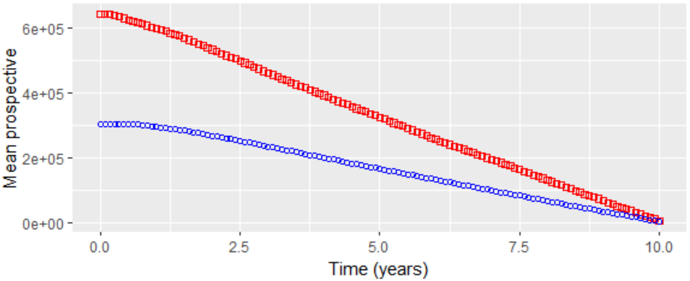

Figure: Mean prospective values of 10 000 generated random insurance cash flows for a policy with disability pension *with* (in red) and *without* (in blue) death benefit. Initial value without death benefit: 303 406.4 .

Example: Endowment (discrete)

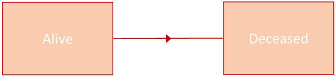

Figure: Survival Markov model.

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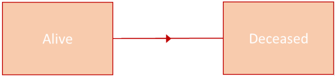

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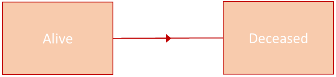
- Transition rates: Finanstilsynet, $\mu(x, t)$, x age and t calendar year and p_{**} is evaluated at discrete times, i.e. $p_{**}(z + n, z + n + 1)$, $n = 0, 1, \ldots$

Example: Endowment (discrete)



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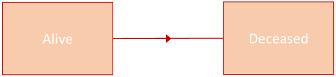

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- Contract duration: $N = 10$ years. Age of insured: $z = 60$ years in 2024.
- **Policy:** If the insured survives to age $z + N = 60 + 10 = 70$, a survival benefit $E = 100\,000$ is paid. If the insured dies during the ages before 70, a death benefit $B = 250\,000$ is paid. Everything stops after $N = 10$ years.

Example: Endowment (discrete)

Policy functions:

For sojourns:

$$a_0(n) = \begin{cases} E, & n = N, \\ 0, & \text{otherwise} \end{cases},$$

For transitions:

$$a_{01}(n) = \begin{cases} B, & n = 1, \dots, N \\ 0, & \text{otherwise} \end{cases}.$$

Comment: $n = 0$ in a_{01} does not make sense, since we assume $Z(0) = 0$ (insured enters contract alive). The earliest a death benefit is assumed to be paid out is thus $n = 1$.

Example: disability pension with death benefit

Recall:

$$\Delta C_k = \sum_{i \in \mathcal{Z}} I_i^Z(k) a_i(k) + \sum_{i,j \in \mathcal{Z}} \Delta N_{ij}^Z(k) a_{ij}(k).$$

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Hence, the prospective value at any time $n = 0, 1, \ldots, N$ is

$$V^+(n) = \frac{E}{v(n)} v(N) I_0^Z(N) + \frac{B}{v(n)} \sum_{k=n+1}^{N} v(k) dN_{01}^Z(k).$$

Example: disability pension with death benefit (continuous)

Prospective value:

$$V^+(n) = \frac{E}{v(n)} v(N) I_0^Z(N) + \frac{B}{v(n)} \sum_{k=n+1}^{N} v(k) \Delta N_{01}^Z(k).$$

- To simulate deterministic cash flows $V^+(n,\omega)$ for a specific outcome ω and times $n = 0, \dots, N$ we need to simulate paths of $Z(n,\omega)$ (states of insured) for a specific outcome ω.

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- Simulate random lives from the function *lifeK2013.R* using the function *life.K13(age,num)* where *age* here is $z = 60$ and *num* is the number of random lives, say 100 or even 10 000!
- For each path ω, you will have determined completely $I_0^Z(N)(\omega)$ and $\Delta N_{01}^Z(k)(\omega)$ then compute $V^+(n,\omega)$ for each ω among the 100 or even 10 000!

Example: disability pension with death benefit (continuous)

Prospective value:

$$V^+(n) = \frac{E}{v(n)} v(N) I_0^Z(N) + \frac{B}{v(n)} \sum_{k=n+1}^{N} v(k) \Delta N_{01}^Z(k).$$

Observe that we can express $V^+(n)$ much faster by using the death time $\tau^z(\omega)$ of each individual ω:

$$V^+(n) = \frac{v(N)}{v(n)} E \mathbb{I}_{\{\tau \geq N\}} \mathbb{I}_{\{n \leq N-1\}} + \frac{v(\tau+1)}{v(n)} B \mathbb{I}_{\{n \leq \tau \leq N-1\}}, \quad n = 0, \ldots, N-1.$$

To find a detailed explanation on the derivation of this expression see Example 4.9 in the lecture notes.

Example: disability pension with death benefit (continuous)

Info: mortality: Finanstilsynet ($G = 0$, $R = 0$). Age $z = 60$. Term of $N = 10$ years. Annual rate $r = 3\%$. Survival benefit $E = 100\,000$ kr. Death benefit $B = 250\,000$ kr.

We generated 1 000 lives with seed 1 and selected four of them, number: 1,2,7 and 18.

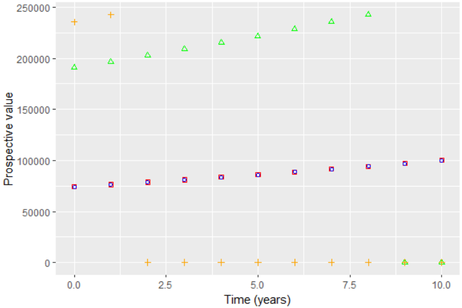

Figure: Four selected prospective values of cash flows from 1 000 randomly generated life paths. Seed: 1. The paths correspond to 1 (in red), 2 (in blue), 7 (in green) and 18 (in orange). We see that 7 and 18 died before N triggering a death benefit.

Example: disability pension with death benefit (continuous)

Remember that we have generated 1 000 paths like those in the previous figure. We may ask, what is the distribution of $V(0)$?

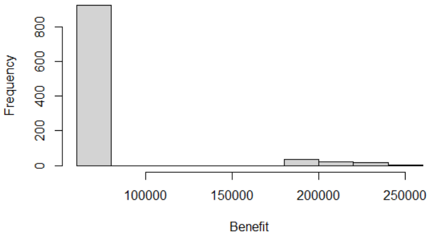

Figure: Histogram of $V(0)$ for 1 000 randomly generated lives.

This random variable will mostly consist of payments of survival benefits of 100 000 in 10 years (hence the lower value 74 081.82) and some payments of 250 000 for those who did not make it to N, discounted according to the time to death.

Example: disability pension with death benefit (continuous)

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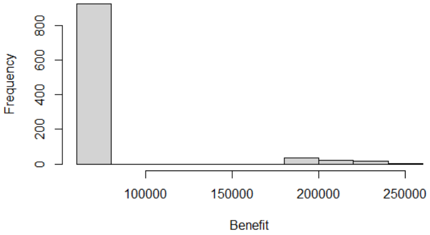

Figure: Histogram of $V(0)$ for 1 000 randomly generated lives.

If we were to charge a premium to all of these individuals it would make sense to charge the expected value of this distribution, i.e. $\mathbb{E}[V(0)]$. In this example, we obtained $\mathbb{E}[V(0)] = 84\,233.05$ kr.

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STK4500: Life insurance and finance

Cash flows and present values