UiO : Department of Mathematics
University of Oslo

STK4500: Life insurance and finance

Expected values and single premium

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Where we are and where we aim at

- Z Markov process with finite state space $\mathcal{Z}$, $p_{ij}(t, s)$, $\mu_{ij}(s)$.

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$$dC(s) = \sum_i I_i^Z(s)da_i(s) + \sum_{i,j:j\neq i} a_{ij}(s)dN_{ij}^Z(s), \quad s \in [0,\infty),$$

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- t-value, retrospective and prospective values:

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- Initial cost (random variable)

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- Expected initial cost (aka single premium π_0) $\pi_0 = \mathbb{E}[V(0)|Z(0) = *]$.
- Aim of this chapter:

$$\text{Compute analytically:} \quad \pi_0 = \mathbb{E}[V(0)], \quad V_i^+(t) = \mathbb{E}[V^+(t)|Z(t) = i].$$

both in discrete and continuous time.

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Computing $V_i^+(t) = \mathbb{E}[V^+(t)|Z(t) = i]$

We have $V^+(t) = \frac{1}{v(t)} \int_{(t,\infty)} v(s)dC(s)$ the prospective value of the future cash flow C after time t.

Computing $V_i^+(t) = \mathbb{E}[V^+(t)|Z(t) = i]$

We have $V^+(t) = \frac{1}{v(t)} \int_{(t,\infty)} v(s)dC(s)$ the prospective value of the future cash flow C after time t.
A meaningful quantity is obviously the **expectation** of this random variable, given the current state of the insured, i.e.

$$V_i^+(t) = \mathbb{E}[V^+(t)|Z(t) = i].$$

Computing $V_i^+(t) = \mathbb{E}[V^+(t)|Z(t) = i]$

We focus on the continuous time case:

$$V^+(t) = \frac{1}{v(t)} \sum_j \int_{(t,\infty)} v(s) I_j^Z(s) da_j(s) + \frac{1}{v(t)} \sum_{j,k:k \neq j} \int_{(t,\infty)} v(s) a_{jk}(s) dN_{jk}^Z(s)$$

Computing $V_i^+(t) = \mathbb{E}[V^+(t)|Z(t) = i]$

We highlight the stochastic parts

$$V^+(t) = \frac{1}{v(t)} \sum_j \int_{(t,\infty)} v(s) I_j^Z(s) da_j(s) + \frac{1}{v(t)} \sum_{j,k:k\neq j} \int_{(t,\infty)} v(s) a_{jk}(s) dN_{jk}^Z(s)$$

Computing $V_i^+(t) = \mathbb{E}[V^+(t)|Z(t) = i]$

So expectation affects only the stochastic parts:

$$\mathbb{E}[V^+(t)|Z(t)=i] = \frac{1}{v(t)}\sum_j \mathbb{E}\left[\int_{(t,\infty)} v(s) I_j^Z(s) da_j(s) \Big| Z(t)=i\right] + \frac{1}{v(t)}\sum_{j,k:k\neq j} \mathbb{E}\left[\int_{(t,\infty)} v(s) a_{jk}(s) dN_{jk}^Z(s) \Big| Z(t)=i\right].$$

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Aim: to compute things of the type

$$\mathbb{E}\left[\int_{(t,\infty)} f(s) I_j^Z(s) da(s) \Big| Z(t) = i \right] \quad \text{and} \quad \mathbb{E}\left[\int_{(t,\infty)} f(s) dN_{jk}^Z(s) \Big| Z(t) = i \right]$$

for functions f and a.

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for functions f and a. This is the easy one.

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Note that for $s > t$,

$$\mathbb{E}[I_j^Z(s)|Z(t) = i] = \mathbb{E}[\mathbb{I}_{\{Z(s)=j\}}|Z(t) = i] = \mathbb{P}[Z(s) = j|Z(t) = i] = p_{ij}(t, s).$$

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Therefore,

$$\mathbb{E}\left[\int_{(t,\infty)} f(s) I_j^Z(s) da(s) \Big| Z(t) = i\right] = \int_{(t,\infty)} f(s)\mathbb{E}[I_j^Z(s)|Z(t) = i] dg(s) = \int_{(t,\infty)} f(s) p_{ij}(t,s) da(s).$$

Computing $V_i^+(t) = \mathbb{E}[V^+(t)|Z(t) = i]$

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Idea: assume just that $f(s) = \mathbb{I}_{[a,b]}(s)$ then the result follows from standard limit theorems from analysis.

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Define

$$g(s) \triangleq \mathbb{E}[N_{jk}^Z(s)|Z(t) = i], \quad s \geq t,$$

the expected number of jumps from j to k in $[0, s]$ given $Z(t) = i$.

Computing $V_i^+(t) = \mathbb{E}[V^+(t)|Z(t) = i]$

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$$\begin{aligned} g(s+h) - g(s) &= \mathbb{E}[N_{jk}^X(s+h) - N_{jk}^Z(s)|Z(t) = i] \\ &= \sum_l \mathbb{E}[\mathbb{I}_{\{Z(s)=l\}}(N_{jk}^Z(s+h) - N_{jk}^Z(s))|Z(t) = i] \end{aligned}$$

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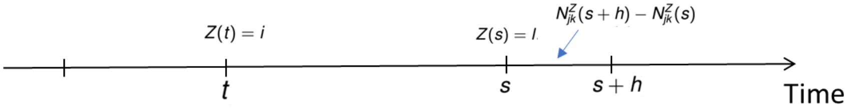
Computing $V_i^+(t) = \mathbb{E}[V^+(t)|Z(t) = i]$

$N_{jk}^Z(s+h) - N_{jk}^Z(s)$ number of jumps from j to k in $[s, s+h]$ is independent of $Z(t) = i$, given $Z(s) = l$.



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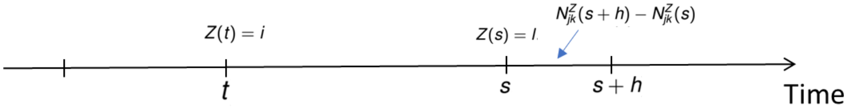

Hence,

$$\begin{aligned}\mathbb{E}[(N_{jk}^Z(s+h) - N_{jk}^Z(s))\mathbb{I}_{\{Z(t)=i\}}|Z(s) = l] = \\ &= \mathbb{E}[(N_{jk}^Z(s+h) - N_{jk}^Z(s))|Z(s) = l]\mathbb{E}[\mathbb{I}_{\{Z(t)=i\}}|Z(s) = l] \\ &= \mathbb{E}[(N_{jk}^Z(s+h) - N_{jk}^Z(s))|Z(s) = l]\mathbb{P}[Z(t) = i|Z(s) = l] \\ &= \mathbb{E}[(N_{jk}^Z(s+h) - N_{jk}^Z(s))|Z(s) = l]\frac{\mathbb{P}[Z(t) = i, Z(s) = l]}{\mathbb{P}[Z(s) = l]}.\end{aligned}$$

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Observe that

$$
\Delta(h) \triangleq \mathbb{E}[(N_{jk}^X(s+h) - N_{jk}^X(s))|X_s = l] = o(h), \text{ for all } l \neq j.
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Taking into account that $Z(t,\omega)$ is right-continuous with left limits and $\mathcal{Z}$ finite we find that

$$\frac{\Delta(h)}{h} \xrightarrow{h \searrow 0} \begin{cases} \mu_{lk}(s), & \text{if } l = j, \\ 0, & \text{else} \end{cases} .$$

Computing $V_i^+(t) = \mathbb{E}[V^+(t)|Z(t) = i]$

Hence,

$$g'(s) = \lim_{h\to 0} \frac{g(s+h) - g(s)}{h} = p_{ij}(t,s)\mu_{jk}(s).$$

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$$g(b) - g(a) = \int_a^b g'(s)ds = \int_a^b p_{ij}(t,s)\mu_{jk}(s)ds = \int_{[0,\infty)} f(s)p_{ij}(t,s)\mu_{jk}(s)ds.$$

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$$g(b) - g(a) = \mathbb{E}[N_{jk}^X(b) - N_{jk}^X(a)|X_t = i] = \mathbb{E}\left[\int_a^b dN_{jk}^X(s)|X_t = i\right]$$
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for $f(s) = \mathbb{I}_{[a,b]}(s)$. Hence, for linear combinations of f as well and by a density argument, for integrable functions.

Recap and conclusion

Aim: to compute things of the type

$$\mathbb{E}\left[\int_{(t,\infty)} f(s) I_j^Z(s) da(s) \middle| Z(t) = i\right] \quad \text{and} \quad \mathbb{E}\left[\int_{(t,\infty)} f(s) dN_{jk}^Z(s) \middle| Z(t) = i\right]$$

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$$\mathbb{E}\left[\int_{(t,\infty)} f(s) I_j^Z(s) da(s) \Big| Z(t) = i\right] = \int_{(t,\infty)} f(s) p_{ij}(t,s) da(s),$$

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Compare with:

$$\mathbb{E}[V^+(t)|Z(t)=i] = \frac{1}{v(t)} \sum_j \mathbb{E}\left[\int_{(t,\infty)} v(s) I_j^Z(s) da_j(s) \middle| Z(t)=i\right] + \frac{1}{v(t)} \sum_{j,k:k\neq j} \mathbb{E}\left[\int_{(t,\infty)} v(s) a_{jk}(s) dN_{jk}^Z(s) \middle| Z(t)=i\right].$$

Recap and conclusion

$$\mathbb{E}\left[\int_{(t,\infty)} f(s) I_j^Z(s) da(s) \Big| Z(t) = i\right] = \int_{(t,\infty)} f(s) p_{ij}(t,s) da(s),$$

$$\mathbb{E}\left[\int_{(t,\infty)} f(s) dN_{jk}^Z(s) \Big| Z(t) = i\right] = \int_{(t,\infty)} f(s) p_{ij}(t,s) \mu_{jk}(s) ds.$$

Compare with:

$$\mathbb{E}[V^+(t)|Z(t)=i] = \frac{1}{v(t)} \sum_j \mathbb{E}\left[\int_{(t,\infty)} v(s) I_j^Z(s) da_j(s) \Big| Z(t)=i\right] + \frac{1}{v(t)} \sum_{j,k:k\neq j} \mathbb{E}\left[\int_{(t,\infty)} v(s) a_{jk}(s) dN_{jk}^Z(s) \Big| Z(t)=i\right].$$

$$\mathbb{E}[V^+(t)|Z(t)=i] = \frac{1}{v(t)} \sum_j \int_{(t,\infty)} v(s) p_{ij}(t,s) da_j(s) + \frac{1}{v(t)} \sum_{j,k:k\neq j} \int_{(t,\infty)} v(s) a_{jk}(s) p_{ij}(t,s) \mu_{jk}(s) ds.$$

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In a summary,

$$\boxed{V_i^+(t) = \frac{1}{v(t)} \sum_j \int_{(t,\infty)} v(s) p_{ij}(t,s) da_j(s) + \frac{1}{v(t)} \sum_{j,k:k\neq j} \int_{(t,\infty)} v(s) a_{jk}(s) p_{ij}(t,s) \mu_{jk}(s) ds}$$

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Computing $\mathbb{E}[V(0)|Z(0) = i]$

It is quite immediate to deduce that $V_i(0) = \mathbb{E}[V(0)|Z(0) = i]$ has the following analytic expression

$$\pi_0 \triangleq V_i(0) = \sum_j \int_{[0,\infty)} v(s) p_{ij}(0,s) da_j(s) + \sum_{j,k:k\neq j} \int_{[0,\infty)} v(s) a_{jk}(s) p_{jk}(0,s) \mu_{jk}(s) ds$$

which corresponds to the **single premium** or **premium paid upfront** of the policy given that the insured enters the contract in state i. Usually $i = *$.

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Expected prospective value and single premium in discrete time

The expected prospective value for $n = 0, 1, \dots$ is given by

$$V_i^+(n) = \frac{1}{v(n)} \sum_j \sum_{m=n+1}^{\infty} v(m) p_{ij}(n, m) a_j(m) + \frac{1}{v(n)} \sum_{j,k} \sum_{m=n+1}^{\infty} v(m) p_{ij}(n, m-1) p_{jk}(m-1, m) a_{jk}(m)$$

and the single premium π_0 by

$$\pi_0 = V_i(0) = \sum_j \sum_{m=0}^{\infty} v(m) p_{ij}(0, m) a_j(m) + \sum_{j,k} \sum_{m=1}^{\infty} v(m) p_{ij}(0, m-1) p_{jk}(m-1, m) a_{jk}(m).$$

Exercise: prove the formulas. Take $\mathbb{E}[\cdot | Z_n = i]$ on the discrete version of the prospective value $V^+(n)$. Use the law of total probability with $\mathbb{I}_{\{Z_{m-1} = l\}}$ and use the Markov property. In other words: follow the steps as for the continuous time case. Here, the proof is easier.

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