UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Constituent exam in: STK4500/9500 — Life Insurance and Finance

Day of examination: 6th June 2023

Examination hours: 9:00 am – 1:00 pm

This problem set consists of 3 pages.

Appendices: None

Permitted aids: Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

This exam consists of four problems. The first one is of theoretical nature, while the three remaining problems are more applied. Make sure to be **precise** and **rigorous** when stating mathematical results and formulae. Please, **write clearly**, **orderly** and avoid crossing out.

Problem 1 Theory (2 points)

State and prove Thiele's difference equation (discrete time setting).

Problem 2 Surplus (3 points)

Let A be a cash flow. We define the so-called *retrospective value* of A as

$$V_t^- \triangleq \frac{1}{v(t)} \int_0^t v(s)dA(s), \quad t \geq 0,$$

where $v(t) = e^{-\int_0^t r(s)ds}$ is a discount factor with interest rate $r(t)$, $t \geq 0$.

The quantity V_t^- represents the t-value of the past benefits minus premiums compounded with r. In reality, the insurer uses a very prudent choice of r, hereby denoted by $\widehat{r}$, to price their policies, to avoid big losses. This often generates a so-called *surplus* which is defined as

$$Q_t \triangleq V_t^- - \widehat{V}_t^-,$$

where $\widehat{V}_t^-$ is V_t^- with $\widehat{r}$ in place of r. The above quantity describes the difference between the assumed and the realized interest rate. Obviously, $Q_0 = 0$, which means that there is no surplus at the beginning of the policy.

(a) (1p) Prove that the surplus process Q satisfies the following ordinary differential equation and give an interpretation of it.

$$\frac{dQ_t}{dt} = r(t)Q_t + (r(t) - \widehat{r}(t))\widehat{V}_t^-, \quad Q_0 = 0.$$

(Continued on page 2.)

(b) (0.5p) In view of the above fact, show that

$$Q_t = \frac{1}{v(t)} \int_0^t v(s)(r(s) - \widehat{r}(s))\widehat{V}_s^- ds, \quad t \geq 0.$$

(c) (1p) Consider a term insurance with death benefit B. You have priced this insurance using a constant interest rate $\widehat{r} > 0$ which gave a single premium of π_0 units, but the actual interest rate turned out to be $r(t) = \widehat{r} + c$ with $c > 0$. Show that the surplus in this case is given by

$$Q_t = -\pi_0 e^{\widehat{r}t}\left(e^{ct} - 1\right) + Be^{\widehat{r}(t-\tau)}\left(e^{c(t-\tau)} - 1\right)\mathbb{I}\{\tau < t\},$$

where τ is the (random) death time of the insured. Explain the result.

(d) (0.5p) In the setting of the previous item, compute the quantity

$$\mathbb{E}[Q_t | X_t = *],$$

where X_t denotes the state of the insured at time t. Provide an interpretation of the result.

Problem 3 The winners take it all (3 points)

Consider a group of N friends with independent lives and equal mortality. Let Z_t be the stochastic process counting the number of friends alive at time t. Assume $Z_0 = N$ and denote by $p(t,s)$ the individual survival probability between times t and s.

An insurance company offers this group of friends the following contract: those who survive till time T will receive a benefit of B monetary units to be shared among them. The insurer uses a discount factor $v(t)$, $t \geq 0$ to value their liabilities.

Let V_t^+ be the prospective value of this policy and $V_n^+(t)$ the expected prospective value, given that there are n living friends at time t.

(a) (0.5p) Argue that for every t, s such that $0 \leq t \leq s$ and $m = 0, \ldots, n$,

$$p_{nm}(t,s) \triangleq \mathbb{P}[Z_s = m | Z_t = n] = \binom{n}{m} p(t,s)^m (1 - p(t,s))^{n-m}$$

and comment briefly on why $p_{nm}(t,s) = 0$ for all $m > n$.

(b) (1p) Specify the policy functions of this policy and show, using the definition of prospective value, that

$$V_t^+ = \frac{v(T)}{v(t)} B\mathbb{I}_{\{Z_T \neq 0\}}, \quad t \in [0, T],$$

where $\mathbb{I}_A$ denotes the indicator function on a set A.

(Continued on page 3.)

(c) (0.5p) What is the probability that this policy will not cost the insurer anything, given that there are still n survivors at time t? What happens when $n \to \infty$? Why?

(d) (1p) Show that

$$V_n^+(t) = \frac{v(T)}{v(t)} B \left(1 - (1 - p(t,T))^n\right)$$

and comment briefly on the formula. What is the price that each participant should pay for such a policy?

Problem 4 Unit-linked policy (2 points)

Let S_t denote the value of a risky fund at time t with constant volatility σ (Black-Scholes model) and r be a constant interest rate. Consider an option on S_T paying the quantity $H(S_T)$ at maturity time T where $H : \mathbb{R} \to \mathbb{R}$ is a given function.

Consider an insurance policy with two states $\mathscr{S} = \{*, \dagger\}$ linked to the fund S with the following specifications

$$\Delta a_*(T) = f_*(T, S_T) = H(S_T), \quad a_{*\dagger}(t) = h_{*\dagger}(t, S_t) = V(t, S_t),$$

where $(t, x) \mapsto V(t, x)$ denotes the expected prospective value of this insurance at time t when the insured is alive at time t and the level of the fund S_t is x. All other benefits are 0 and we consider no premiums.

(a) (0.5 p) Write down the partial differential equation for V and the final condition. You can assume that the boundary conditions are settled, i.e. $V(t, 0) = b_1(t)$ and $V(t, L) = b_2(t)$ for given functions b_1 and b_2 and cut-off value L.

(b) (1.5p) Let $\{(t_i, x_j)\}_{\substack{i=1,\ldots,n \\ j=1,\ldots,m}}$ be a uniform partition of $[0, T] \times [0, L]$ where T is the maturity time of the policy and L some large cut-off value of the fund and Δt and Δx denote the distance between time and space points, respectively. Let V_i^j be an approximation of the solution $V(t_i, x_j)$ at the point (t_i, x_j). Detail how you would find V_i^j using the implicit method.

Assuming that $S(0) = x_{j^*}$ for some $j^* = 0, \ldots, m$, what would then be the single premium of this policy?

GOOD LUCK!

(Continued on page 4.)