UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Deferred exam in: STK4500/9500 — Life Insurance and Finance

Day of examination: 17th August 2023

Examination hours: 3:00 pm – 7:00 pm

This problem set consists of 3 pages.

Appendices: None

Permitted aids: Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

This exam consists of four problems. The first one is of theoretical nature, while the three remaining problems are more applied. Make sure to be **precise** and **rigorous** when stating mathematical results and formulae. Please, **write clearly**, **orderly** and avoid crossing out.

Problem 1 Theory (2 points)

Answer the following questions:

(a) (0.5p) Define the prospective value V_t^+ of a cash flow A and explain what it describes.

(b) (1p) Define the concept of policy functions and express the prospective value in terms of policy functions.

(c) (0.5p) Define the expected prospective value $V_i^+(t)$ for a given state i and give an explicit formula for computing it in terms of transition probabilities and transition rates.

Problem 2 Until death do them part (3 points)

Consider a married couple living in different countries (so we may assume independent lives for simplicity). They sign a policy with the following specification: if one of them dies, the insurer pays the survivor a pension P (yearly) continuously until death. Denote by T_i, $i = 1, 2$ their remaining life times at the beginning of the contract. Assume

$$T_i \sim \exp(\mu_i), \quad i = 1, 2,$$

that is T_i is exponentially distributed with parameter μ_i for each $i = 1, 2$ (expectation is $1/\mu_i$).

Finally, suppose the force of interest is constant $r > 0$

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(a) (1p) Show that the single premium of this policy is given by

$$\pi_0 = \frac{P}{r}\left[2\frac{\mu_1+\mu_2}{r+\mu_1+\mu_2} - \frac{\mu_1}{r+\mu_1} - \frac{\mu_2}{r+\mu_2}\right].$$

(b) (1p) Prove that the prospective value of this policy is given by

$$V_t^+ = \frac{P}{r}e^{rt}\left[e^{-r(T_{(1)}\vee t)} - e^{-r(T_{(2)}\vee t)}\right],$$

where $T_{(1)}$ denotes the minimum between T_1 and T_2 and $T_{(2)}$ denotes the maximum between T_1 and T_2.

(c) (1p) Partner 1 of this policy calls you (at time t) in grief saying that their partner passed away on a trip swimming with sharks. Based on this information. What is the expected prospective value of the remaining liabilities?

Problem 3 Pensions for the winners (3 points)

Consider a group of N friends with independent lives and equal mortality. Let Z_t, $t \geq 0$, be the stochastic process counting the number of friends alive at time t. Assume $Z_0 = N$ and denote by $p(t,s)$ the individual survival probability between times t and s.

An insurance company offers this group of friends the following contract: those who survive till time T_0 will receive, continuously, a yearly pension of P monetary units to be shared among them, until time $T > T_0$ where the payments stop. The insurer uses a discount factor $v(t)$, $t \geq 0$ to value their liabilities.

Let V_t^+ be the prospective value of this policy and $V_n^+(t)$ the expected prospective value, given that there are n living friends at time t.

(a) (0.5p) Argue that for every t, s such that $0 \leq t \leq s$ and $m = 0, \dots, n$,

$$p_{nm}(t,s) \triangleq \mathbb{P}[Z_s = m | Z_t = n] = \binom{n}{m} p(t,s)^m (1-p(t,s))^{n-m}$$

and comment briefly on why $p_{nm}(t,s) = 0$ for all $m > n$.

(b) (1p) Specify the policy functions of this policy and show, using the definition of prospective value, that

$$V_t^+ = \frac{P}{v(t)}\int_{(t\vee T_0)\wedge\tau_{(N)}}^{(t\vee T)\wedge\tau_{(N)}} v(s)ds, \quad t \in [0,T],$$

where $\tau_{(N)}$ denotes the death time of the last survivor, i.e. the maximum between all death times $\tau_1, \dots, \tau_N$. Note: the symbol $\vee$ means maximum, while $\wedge$ means minimum.

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(c) (0.5p) What is the probability that this policy will not cost the insurer anything, given that there are still n survivors at time t, $t \in [0, T_0]$? What happens when $n \to \infty$? Why?

(d) (1p) Show that

$$V_n^+(t) = \frac{P}{v(t)} \int_{t \vee T_0}^{t \vee T} v(s) \left(1 - (1 - p(t,s))^n\right) ds$$

and comment briefly on the formula. What is the price that each participant should pay for such a policy?

Problem 4 Equity-linked inheritance (2 points)

A parent would like to secure an inheritance for their child. For this reason the parent writes an insurance with you as insurer. You manage a portfolio with a bank account at constant interest r and two independent funds S_1 and S_2. The parent gives you K monetary units to be invested in the bank account and these two funds. The parent wants to invest half of it into the bank account and the rest in equal parts into the funds. If the parent dies, you give the child the value of the investment as inheritance at the time of death. There are no other payments.

(a) (1p) What is the explicit formula for the expected prospective value $V_*^+(t, S_1(t), S_2(t))$ of this policy?

(b) (1p) Assume the following information: age of parent 30 years. Interest rate $r = 3\%$. Funds follow a Black-Scholes model with the following specifications: current value of the funds $S_1(0) = 50€$, $S_2(0) = 20€$ with respective risks $\sigma_1 = 20\%$ and $\sigma_2 = 10\%$. Assume constant mortality $\mu = 0.01$ and an initial investment with capital $K = 100\,000€$. Under these circumstances, what is the single premium to be charged? Can you explain why or what is the reason? Could you explain whether the insurer here is bearing any risk and what would that be?

GOOD LUCK!

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