STK4500/9500: Life Insurance and Finance
Mandatory assignment Spring 2024

Submission deadline

Thursday 9th of May 2024, 14:30 in Canvas (`canvas.uio.no`).

Instructions

The assignment must be submitted as a single pdf file in LaTeX. The submission must contain your name and course code.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. You only have **one attempt** at the assignment, and you need to have the assignment approved in order to take the exam. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. Use of external material or another person's work must be cited. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

IMPORTANT: To pass the assignment you should try solving all items and, ideally, manage to carry out 50% of the assignment correctly. Each item is worth one point.

Complete guidelines about delivery of mandatory assignments:

`uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html`

Problem 1: Mortgage loan insurance

You work in the risk department of a bank and have been entrusted with the task to implement a continuous time model for mortgage loans and assess the risk of default due to demise of the borrower.

A mortgage is a contract between a lender, usually a bank, and a borrower, usually a private person. The loan grants the borrower a fixed amount of money, known as the principal. The borrower agrees to pay back the loan on a monthly basis for a certain period of time. The monthly amounts are known as instalments and the repayment period can be upto several years, for instance, 25 or 30 years. The purpose of the mortgage is to buy a property, which serves as collateral to secure the loan.

We will consider a loan with principal P and monthly instalments of B monetary units. We employ the technical discount factor $v(t) = e^{-rt}$, $t \geq 0$, where r is a constant nominal rate (yearly).

(a) Express the cash flow of a bank loan with principal P transferred today with monthly instalments of equal amounts of B monetary units starting from next month, for a period of T months, that is $T/12$ years. Assuming a constant technical interest rate r, write down the present value $V(t)$, the retrospective value $V^{-}(t)$ and the prospective value $V^{+}(t)$ of the loan.

(b) Let $P = 1\,500\,000$ kr with a repayment period of 20 years (i.e. $T = 240$ months) with 5% yearly interest rate. Compute how much are the monthly instalments B that the borrower must pay back to the bank. Plot the present, retrospective and prospective values for each $t \in [0, T)$. What do you observe? After 7 years, the borrower wishes to cancel the loan and pay back everything left at once. How much would that be?

(c) In reality, most of the mortgages are so-called *adjustable-rate interest mortgages* meaning that the interest rate can change according to some interest rate index. Here, we take as example interest rates for mortgages from December 2021 to December 2023. See the Figure 3 below.

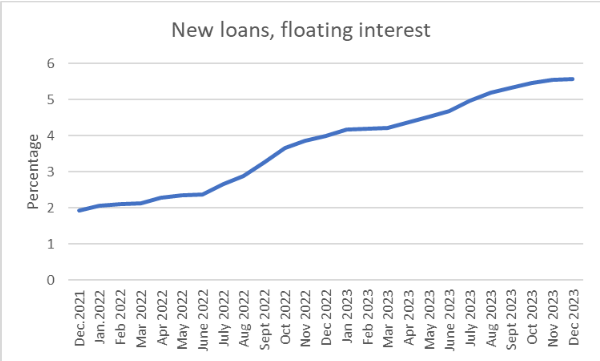

Figure 1: Interest rate applied to new loans issued monthly. Source: SSB.

Interest rate evolution for new loans	
Month	Rate (%)
Dec 2021	1.92
Jan 2022	2.06
Feb 2022	2.10
Mar 2022	2.12
Apr 2022	2.28
May 2022	2.35
June 2022	2.38
July 2022	2.67
Aug 2022	2.88
Sept 2022	3.27
Oct 2022	3.65
Nov 2022	3.87
Dec 2022	3.99
Jan 2023	4.17
Feb 2023	4.19
Mar 2023	4.22
Apr 2023	4.38
May 2023	4.52
June 2023	4.69
July 2023	4.98
Aug 2023	5.19
Sept 2023	5.33
Oct 2023	5.46
Nov 2023	5.54
Dec 2023	5.57

Table 1: Yearly loan rates.

The table shows the interest rate fluctuations for 25 months. Assume for a moment that the loan in item (c) was granted on the 1st of December 2021 and the first instalment is due on the 1st of January 2022. Compute the amount B that is offered to the borrower on the 1st of December 2021, hereby B_0. Thereafter, when the new rate is published on the 1st of January, the amount B_0 is readjusted to a new amount B_1 according to the new interest rate. For the first step we start with $r = 1.92\%$ then it is increased to $r = 2.06\%$ and so on. Compute the new amount B_1 and do the same for the coming months, B_2, $\dots$, B_{24}. You can still asumme that $T = 20$ years. This exercise is just about updating the mortgage instalments according to the interest rate fluctuations.

Compute the final balance that the borrower has paid for these 25 months and the fixed constant rate that would make up to the same balance. This rate, would be the threshold under which a fixed-rate mortgage would be more profitable than an adjustable-rate mortgage.

(d) Assume again that $r = 5\%$ is the employed technical rate by the bank to price loans. Assess the risk of loss due to death of the borrower. Use the mortality basis published by Finanstilsynet in their K2013 letter. You can take your age and legal gender as example. Use mortality risk. For the solution we will use: $z = 30$ in $Y = 2024$ and $G = R = 0$.

More concretely, carry out the following computations:

(i) Estimate the loss distribution of the bank due to death.

(ii) Compute the mean loss analytically and by Monte-Carlo. Compare the results.

(iii) Compute the solvency capital requirement, i.e. the 99.5%-quantile of the loss distribution.

Hint: the loss caused by death is naturally a distribution, whose values differ from person to person. The present value of the outstanding loan at time t is $V^+(t)$ from item (a). If τ denotes the death time of the borrower, then the relevant quantity to look at would be $v(\tau)V^+(\tau)\mathbb{I}_{\{\tau\leq T\}}$, which is the discounted outstanding loan at the time of death where a loss happens only if death occurs before expiration of the loan.

(e) Compute the monthly premium from item (d), i.e. the deferred premium on a monthly basis for that policy. Assume that premiums are paid monthly in advance and the last payments occurs on the first day of the 239th month.

Problem 2: Disability insurance with retirement

We consider a multistate-state Markov model with three states in continuous time, see Figure 2. Z will be the Markov process with states in $\mathcal{Z} = \{0, 1, 2\}$ where 0 is the active/alive state, 1 is the disability state and 2 is the life ending state.

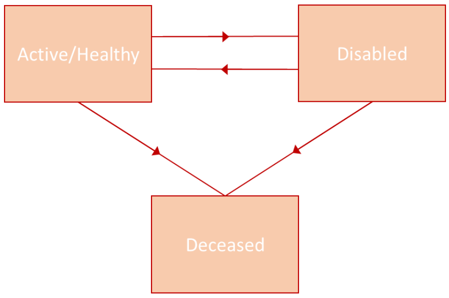

Figure 2: Disability model.

The transition rates that we will assume are contained in the following transition rate matrix

$$\Lambda(t) = \begin{pmatrix} \mu_{00}(t) & \mu_{01}(t) & \mu_{02}(t) \\ \mu_{10}(t) & \mu_{11}(t) & \mu_{12}(t) \\ \mu_{20}(t) & \mu_{21}(t) & \mu_{22}(t) \end{pmatrix}.$$

Biometric specifications: Assume the following transition rates

$$\mu_{01}(x) = 0.0004 + 10^{0.06x-5.46}, \qquad \mu_{02}(x) = 0.0005 + 10^{0.038x-4.12},$$
$$\mu_{10}(x) = 0.05, \qquad \mu_{12}(x) = \mu_{02}(x),$$

where x is age.

Policy specifications: Policyholder is a $z = 30$ year old individual with right to a disability pension of $D = 100\,000$ monetary units per annum as long as they are in the disability state from contract inception until retirement time $T_0 = 40$ years after inception and right to pension of $P = 300\,000$ monetary units per annum from retirement time T_0 to expiry time $T = 80$ as long as they are in the active/alive or disability state.

Economic/financial specification: We assume a technical constant interest rate of $r = 3\%$ per annum.

(a) Find the transition probabilities for this model by solving Kolmogorov's equation numerically.

Then, make a program that simulates random lives, i.e. random paths of $t \mapsto Z(t, \omega)$ where ω is the life of an individual. An example of what a path should look like is given in Figure 3 below.

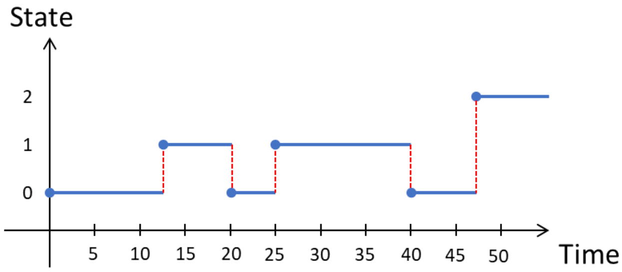

Figure 3: Example of a life performance of an individual.

To simulate lives from this model, you may use the following algorithm:

1. Take a partition of $[0, T]$ of size h and let $t_i = ih$, $i = 0, \dots, n$ where $n = \frac{T}{h}$.
2. Set $Z(0) = 0$.
3. For $i = 1, \dots, n$, generate a random observation, say j_i, from $\{0, 1, 2\}$ with the probabilities $\{p_{Z(t_{i-1}),0}(t_{i-1}, t_i), p_{Z(t_{i-1}),1}(t_{i-1}, t_i), p_{Z(t_{i-1}),2}(t_{i-1}, t_i)\}$. Set $Z(t_i) = j_i$.

After completing the algorithm you will have a vector of $n + 1$ states, one for each time point t_i, $i = 0, \dots, n$ and where $Z(0) = 0$. Plot two, three or four random paths in one figure.

(b) Write down the policy functions of this insurance policy, the policy cash flow C described by the policy functions and the associated present, retrospective and prospective values of the policy cash flow.

(c) Plot the expected prospective values $V_0^+(t)$ and $V_1^+(t)$ for $t \in [0, T]$ and explain what these functions mean.

(d) Simulate a histogram of $V(0, C)$ using the algorithm in (a). Compute the empirical mean of $V(0, C)$ and the single premium using the exact formula from the lecture notes. Compare and discuss. Compute the yearly premiums of this insurance assuming that the insured pays premiums during the working period only if they are in the active state and plot the function $t \mapsto V_*^+(t, C) + V_*^+(t, C^\pi)$, where C^π is the cash flow associated to the payment of premiums.

(e) Consider now the case where a death benefit is paid out corresponding to the expected prospective value of the policy at death time (refund guarantee). Recompute the single premium of this policy when adding such benefit to the contract. Comment.