STK4500/9500: Life Insurance and Finance Mandatory assignment Spring 2024

Submission deadline

Thursday 9th of May 2024, 14:30 in Canvas (`canvas.uio.no`).

Instructions

The assignment must be submitted as a single pdf file in LaTeX. The submission must contain your name and course code.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. You only have **one attempt** at the assignment, and you need to have the assignment approved in order to take the exam. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. Use of external material or another person's work must be cited. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

IMPORTANT: To pass the assignment you should try solving all items and, ideally, manage to carry out 50% of the assignment correctly.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

Problem 1: Mortgage loan insurance

You work in the risk department of a bank and have been entrusted with the task to implement a continuous time model for mortgage loans and assess the risk of default

due to demise of the borrower.

A mortgage is a contract between a lender, usually a bank, and a borrower, usually a private person. The loan grants the borrower a fixed amount of money, known as the principal. The borrower agrees to pay back the loan on a monthly basis for a certain period of time. The monthly amounts are known as instalments and the repayment period can be upto several years, for instance, 25 or 30 years. The purpose of the mortgage is to buy a property, which serves as collateral to secure the loan.

We will consider a loan with principal P and monthly instalments of B monetary units. We employ the technical discount factor $v(t) = e^{-rt}$, $t \geq 0$, where r is a constant nominal rate (yearly).

(a) Express the cash flow of a bank loan with principal P transferred today with monthly instalments of equal amounts of B monetary units starting from next month, for a period of T months, that is $T/12$ years. Assuming a constant technical interest rate r, write down the present value $V(t)$, the retrospective value $V^-(t)$ and the prospective value $V^+(t)$ of the loan.

Solution: The basic time unit t will be months. You could have chosen $t = 1$ one year and $t = 1/12$ a month as well. It does not matter as long as you use the right interest rate. For instance, if you choose $t = 1$ monthly, then $r = 5\%/12$ is the right rate to use, whereas if you use $t = 1$ a year, then you can use $r = 5\%$ and in your formulas it will appear $r/12$ automatically. The point is that the basic time unit for t should match the basic time unit for which r is given. Therefore, we will use $r = 5\%/12$.

The cash flow is given by

$$
C(t) = \begin{cases}
P, & t \in [0, 1), \\
P - B, & t \in [1, 2), \\
\dots, \\
P - Bk, & t \in [k, k+1), \\
\dots, \\
P - B(T-1), & t \in [(T-1), T), \\
P - BT, & t \in [T, \infty).
\end{cases}
$$

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If you chose t to be in units of years then C^{yearly} is given by

$$
C^{\text{yearly}}(t) = \begin{cases} P, & t \in [0, h), \\ P - B, & t \in [h, 2h), \\ \ldots, \\ P - Bk, & t \in [kh, (k+1)h), \\ \ldots, \\ P - B(n-1), & t \in [(n-1)h, nh), \\ P - Bn, & t \in [T, \infty), \end{cases}
$$

where $h = 1/12$ is one month and $n = T/h$ the number of months.

This cash flow is a (decreasing) step function with jumps at times $t \in \{0, 1, \ldots, T\}$, a total of $T + 1$ jumps with $\Delta C(0) = P$ and $\Delta C(k) = -B$ for all $k = 1, \ldots, T$.

The present value, retrospective and prospective values are defined by

$$
V(t) = \frac{1}{v(t)} \int_{[0,\infty)} v(s) dC(s), \quad t \geq 0
$$

and

$$
V^-(t) = \frac{1}{v(t)} \int_{[0,t]} v(s) dC(s), \quad V^+(t) = \frac{1}{v(t)} \int_{(t,\infty)} v(s) dC(s), \quad t \in [0, T].
$$

The Riemann-Stieltjes integral with $C'(s) = 0$ where C is differentiable and has $T + 1$ jumps is given by

$$
V(t) = \frac{P}{v(t)} - \frac{B}{v(t)} \sum_{k=1}^{T} v(k).
$$

The expression above makes sense. It is the t-value of the principal P and the t-value of each payment of B monetary units executed at each future month k.

In a similar fashion we can compute $V^-(t)$ and $V^+(t)$. We have,

$$
V^-(t) = \frac{P}{v(t)} - \frac{B}{v(t)} \sum_{k=1}^{T} v(k) \mathbb{I}_{[0,t]}(k) = \frac{P}{v(t)} - \frac{B}{v(t)} \sum_{k=1}^{\lfloor t \rfloor} v(k), \quad t \in [0, T]
$$

and

$$
V^+(t) = -\frac{B}{v(t)} \sum_{k=1}^{T} v(k) \mathbb{I}_{(t,\infty)}(k) = -\frac{B}{v(t)} \sum_{k=\lfloor t \rfloor + 1}^{T} v(k), \quad t \in [0, T].
$$

(b) Let $P = 1\,500\,000$ kr with a repayment period of 20 years (i.e. $T = 240$ months) with 5% yearly interest rate. Compute how much are the monthly instalments B that the borrower must pay back to the bank. Plot the present, retrospective and prospective values for each $t \in [0, T)$. What do you observe? After 7 years, the borrower wishes to cancel the loan and pay back everything left at once. How much would that be?

Solution: The time variable t is given in month units, hence, the constant rate for a month is $r = 5\%/12 = 0.4166\%$. To make the cash flow even, we need to impose that today's value is 0, i.e.

$$V(0) = 0.$$

This gives the equation

$$P = B \sum_{k=1}^{T} v(k).$$

Since $v(k) = e^{-rk}$ then we have a geometric series. So,

$$\sum_{k=1}^{T} e^{-rk} = \frac{1 - e^{-rT}}{e^r - 1},$$

where $r = 5\%/12$ is the monthly rate.

As a result,

$$B = Pe^{r*T} \frac{e^r - 1}{e^{rT} - 1} = 9\,907.98\,\text{kr}.$$

R code for this item:

```
P <- 1500000
T <- 12*20
r <- 0.05/12
v <- function(k) exp(-r*k)
B <- P/((v(1)-v(T+1))/(1-v(1)))
```

The monthly payments were chosen so that the present value of the cash flow is zero, i.e.

$$V(0) = \int_{[0,\infty)} v(s) dC(s) = 0,$$

which implies that

$$V(t) = \frac{1}{v(t)} V(0) = 0$$

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for all $t \in [0, T)$.

Then since $V^-(t) + V^+(t) = V(t) = 0$ we see that $V^-(t)$ and $V^+(t)$ are mirrored quantities. This makes sense. The value $V^+(t)$ is what the borrower owes the bank at time t. This quantity is negative since we adopted the widespread convention that benefits have positive sign and payments/premiums have negative sign. In other words, cash from bank/insurer to customer is positive and cash from customer to bank/insurer has negative sign.
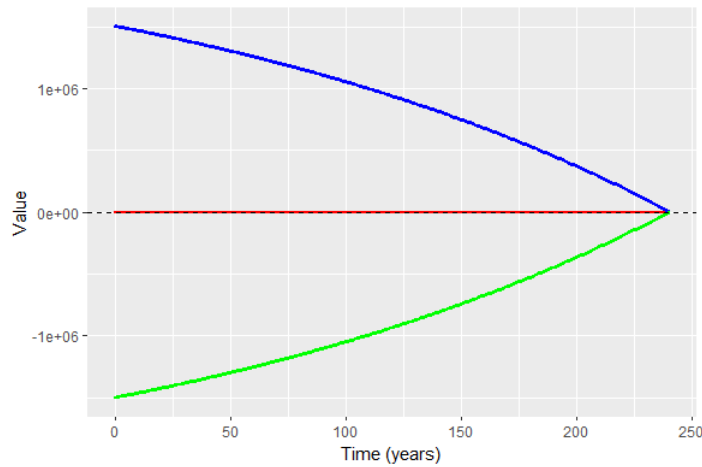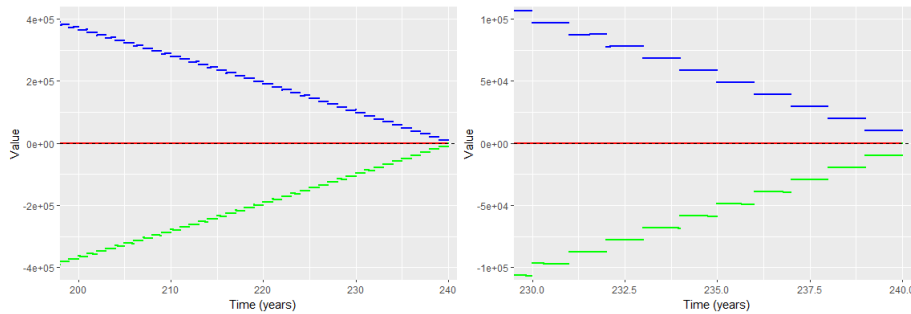

Figure 1: Present value is always 0 (in red). Retrospective value (in blue) and prospective value (in green).

It is not possible to appreciate in Figure 1, but the green and blue curves are actually discontinuous with jumps every month. Just to see it, we zoomed in the figure around the last 40 and 10 months of the loan, respectively.

(a) The last 40 months. (b) The last 10 months.

Figure 2: Present value (in red), retrospective value (in blue) and prospective value (in green) during the last months of the period. We can appreciate a discontinuity at each monthly instalment.

R code for this item:

```
PV <- function(t) P/v(t)-(1/v(t))*B*((v(1)-v(T+1))/(1-v(1)))
rV <- function(t) P/v(t)-(1/v(t))*B*((v(1)-v(floor(t)+1))/(1-v(1)))
pV <- function(t) -(1/v(t))*B*((v(floor(t)+1)-v(T+1))/(1-v(1)))

time <- seq(0,T,by=0.1)

PV.val <- sapply(time,PV)
rV.val <- sapply(time,rV)
pV.val <- sapply(time,pV)

ggplot()+
  geom_point(aes(x = time, y = PV.val),color = "red", size=0.1)+
  geom_point(aes(x = time, y = rV.val),color = "blue", size=0.1)+
  geom_point(aes(x = time, y = pV.val),color = "green", size=0.1)+
  xlab("Time (years)")+ylab("Value")+geom_hline(yintercept=0,
      linetype="dashed")
```

Seven years corresponds to $7 \cdot 12 = 84$ months, i.e. 84 payments. Hence, the outstanding loan would be $V^+(7 \cdot 12) = -1\,134\,169$ kr.

(c) In reality, most of the mortgages are so-called *adjustable-rate interest mortgages* meaning that the interest rate can change according to some interest rate index. Here, we take as example interest rates for mortgages from December 2021 to December 2023. See the Figure 9 below.
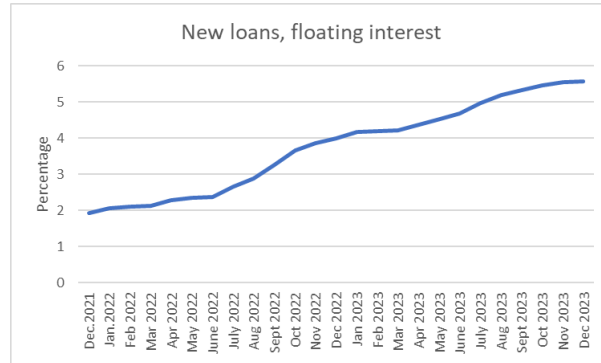
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Figure 3: Interest rate applied to new loans issued monthly. Source: SSB.

Interest rate evolution for new loans	
Month	Rate (%)
Dec 2021	1.92
Jan 2022	2.06
Feb 2022	2.10
Mar 2022	2.12
Apr 2022	2.28
May 2022	2.35
June 2022	2.38
July 2022	2.67
Aug 2022	2.88
Sept 2022	3.27
Oct 2022	3.65
Nov 2022	3.87
Dec 2022	3.99
Jan 2023	4.17
Feb 2023	4.19
Mar 2023	4.22
Apr 2023	4.38
May 2023	4.52
June 2023	4.69
July 2023	4.98
Aug 2023	5.19
Sept 2023	5.33
Oct 2023	5.46
Nov 2023	5.54
Dec 2023	5.57

Table 1: Yearly loan rates.

The table shows the interest rate fluctuations for 25 months. Assume for a moment that the loan in item (c) was granted on the 1st of December 2021 and the first instalment is due on the 1st of January 2022. Compute the amount B that is offered to the borrower on the 1st of December 2021, hereby B_0. Thereafter, when the new rate is published on the 1st of January, the amount B_0 is readjusted to a new amount B_1 according to the new interest rate. For the first step we start with $r = 1.92\%$ then it is increased to $r = 2.06\%$ and so

on. Compute the new amount B_1 and do the same for the coming months, B_2, ..., B_{24}. You can still assume that $T = 20$ years. This exercise is just about updating the mortgage instalments according to the interest rate fluctuations.

Compute the final balance that the borrower has paid for these 25 months and the fixed constant rate that would make up to the same balance. This rate, would be the threshold under which a fixed-rate mortgage would be more profitable than an adjustable-rate mortgage.

Solution:

In this part of the exercise we look at a non-constant interest rate curve r, i.e. $r(t) = r_i$ for $t \in [i, i+1)$, for $i = 0, \ldots, 24$, where r_i is the rate in the table starting from $r_0 = 1.92\%$.

The principle used to find the constant instalments B is equivalence, i.e. one needs to impose that
$$V(0) = 0$$
giving rise to the equation
$$V(0) = P - B \sum_{k=1}^{T} v(k) = 0.$$

The present value needs to be zero at any time, since $V(t) = \frac{V(0)}{v(t)} = 0$. The issue here is that once we apply the equivalence principle for $r_0 = 1.92\%$ to obtain $B_0 = 9\,907.98$ kr then after one month the new rate destroys the equivalence so that $V(1) \neq 0$ (if r increases then the bank loses and vice versa). Hence, future instalments must be updated to respect equivalence of the outstanding loan.

Since we now have a different rate r_1 to be used. We need thus to split the cash flow into the *retrospective value* $V^-(1, r_0)$, where the old rate r_0 was used and the prospective value $V^+(1, r_1)$, where the new rate is used. Then, to find the new instalments B_1 we need to impose that
$$V(1) = \underbrace{V^-(1, r_0)}_{\substack{\text{retrospective cash} \\ \text{with old rate}}} + \underbrace{V^+(1, r_1)}_{\substack{\text{prospective cash} \\ \text{with new rate}}} = 0.$$

This is equivalent to saying that
$$\underbrace{\frac{P}{v(1; r_0)} - B_0}_{= V^-(1, r_0)} = \underbrace{\frac{B_1}{v(1; r_1)} \sum_{k=2}^{T} v(k; r_1)}_{= -V^+(1, r_1)} \left(= B_1 \sum_{k=1}^{T-1} v(k; r_1) \right), \tag{1}$$

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where $v(k; r)$ denotes the discount factor with rate r. Note that the left-hand side of (1) is the paid-in part, where the loan P has been compounded with one month factor $\frac{1}{v(1;r_0)}$ w.r.t. the old interest rate and the right-hand side is the future (new) payments that are now evaluated w.r.t. the new interest rate $r_1 = 2.06\%$.

Now, on the 1st of January, we have a «new» loan with principal $P_1 = \frac{P}{v(1;r_0)} - B_0$ and rate r_1 with instalments of size B_1 to be paid in $T-1$ months.

Iterating this argument, the instalment B_2 should fulfil

$$\frac{P_1}{v(1;r_1)} - B_1 = B_2 \sum_{k=1}^{T-2} v(k; r_2).$$

Defining

$$P_i = \frac{P_{i-1}}{v(1;r_{i-1})} - B_{i-1}, \quad P_0 = P$$

then, assuming that we know B_{i-1}, we can find the updated instalment B_i by solving

$$P_i = B_i \sum_{k=1}^{T-i} v(k; r_i).$$

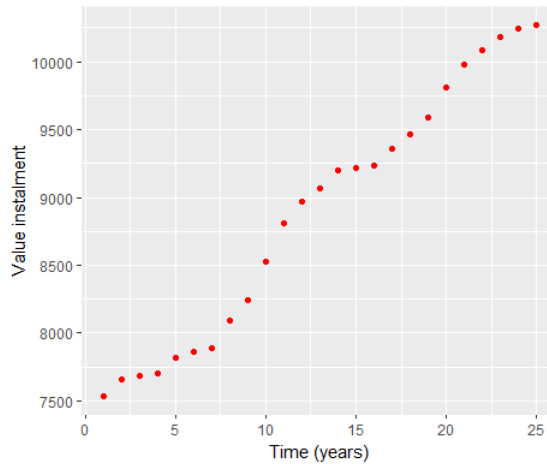We obtained the following results:

Figure 4: Series of updated instalments according to the new published nominal rates for the given 25 months (from 1st Dec. 21 to 1st Dec. 23).

We see that the increase is most prominent between the 8th and the 11th months, which coincides with the highest interest rate increases.

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The question on whether which is the (constant) interest rate for which (constant) instalments match the variable ones is not entirely uniquely determined. It depends on how much monthly B we are willing to pay.

The total balance paid for the (variable) instalments $B_0, \ldots, B_{24}$ is given by the retrospective value $V^-(24)$ at the 25th month. Careful here, because the discount factor $v(t) = e^{-\int_0^t r(s)ds}$ is not constant and $r(s)$ is defined by

$$r(s) = \sum_{k=0}^{\infty} r_k \mathbb{I}_{[k,k+1)}(s),$$

where $r_k = r_{24}$ for all $k \geq 25$, where r_{24} is the last interest rate we know of. Hence,

$$V^-(24) = \frac{P}{v(24)} - \frac{1}{v(24)} \sum_{k=0}^{24} B_k v(k),$$

where we used the formula for the retrospective value with $t = 24$. So, today's value of the paid-in cash is

$$v(24)V^-(24) = P - \sum_{k=0}^{24} B_k v(k).$$

Now, Today's value of constant instalments B for 24 months at constant interest rate is

$$P - B \sum_{k=0}^{24} e^{-rk}.$$

We thus need to match

$$v(24)V^-(24) = P - B \sum_{k=0}^{24} e^{-rk}$$

which gives

$$P - \sum_{k=0}^{24} B_k v(k) = P - B \sum_{k=0}^{24} e^{-rk}$$

and thus

$$\sum_{k=0}^{24} B_k v(k) = B \frac{1 - e^{-25 \cdot r}}{1 - e^{-r}}.$$

The left-hand side above is a number, say b, while the right-hand side depends on the simultaneous choice of B and r.

The number b turned out to be: $b = 150\,709.7$ which is today's value of the paid in $B_0, \ldots, B_{24}$ for the given interest rate table. Then the relation between B and r matching the same value is given in the next figure.
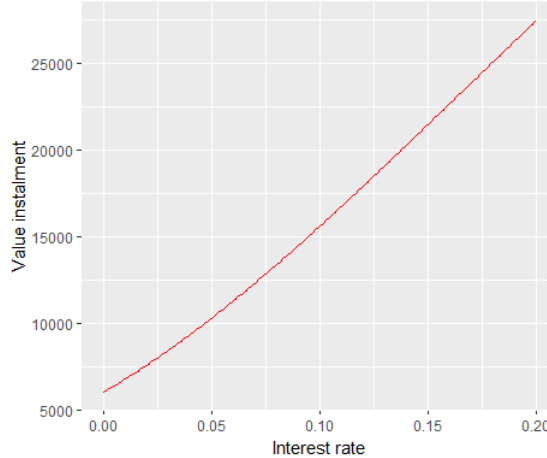

Figure 5: For each given rate, we see the value of (constant) B whose present value matches the present value for the variable rate loan, which was $150\,709.7$. For instance, for constant monthly $B = \frac{1}{25}\sum_{k=0}^{24} B_k = 8\,898.84$ the rate is $r = 0.0351 = 3.51\%$, which is not the mean of the rates in Table 1.

If the borrower could guess the future interest rate evolution, i.e. Table 1 and, in particular, the starting rate 1.92% and they were willing to pay a constant monthly amount of $8\,898.84$ kr., then they should choose a fixed-rate mortgage if the fixed rate was at most 3.51%.

We used the following code:

```
rk <- c(0.0192, 0.0206, 0.0210, 0.0212, 0.0228, 0.0235, 0.0238,
    0.0267, 0.0288, 0.0327, 0.0365, 0.0387, 0.0399, 0.0417, 0.0419,
    0.0422, 0.0438, 0.0452, 0.0469, 0.0498, 0.0519, 0.0533, 0.0546,
    0.0554, 0.0557)
vr <- function(r,t) exp(-(r/12)*t)
Bvec <- rep(0,length(rk))
Pvec <- B
Pvec[1] <- P
Bvec[1] <- Pvec[1]/( (vr(rk[1],1)-vr(rk[1],T+1))/(1-vr(rk[1],1)))
for(j in 2:25){
  Pvec[j] <- Pvec[j-1]/vr(rk[j-1],1) - Bvec[j-1]
  aux <- (vr(rk[j],1)-vr(rk[j],T-j+1))/(1-vr(rk[j],1))
  Bvec[j] <- Pvec[j]/aux
```

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```
}

ggplot()+ geom_point(aes(x =1:25, y =Bvec),color = "red") +
    xlab("Time (years)")+ylab("Value instalment")

#Relation between (constant) B and (constant) r

#Total balance for the B_k and r_k:
bal <- sum(Bvec*exp(-cumsum(rk)))

#We plot (r,B) such that bal=B*((1-exp(-25*r))/(1-exp(-r)))
fun.B <- function(r) bal*((1-exp(-r))/(1-exp(-25*r)))

r.val <- seq(0,0.2,by=0.001) #We just look up to 20% rate, higher is
    overkill
B.val <- sapply(r.val,fun.B)
B.val[1] <- bal/25
ggplot()+ geom_line(aes(x =r.val, y =B.val),color = "red") +
    xlab("Interest rate") + ylab("Value instalment")

#We find the rate which correponds to a B equal to the mean of B_k
f1 <- function(r) mean(Bvec)-bal*((1-exp(-r))/(1-exp(-25*r)))
r.fixed <- uniroot(f1,c(0.0001,1))$root
```

(d) Assume again that $r = 5\%$ is the employed technical rate by the bank to price loans. Assess the risk of loss due to death of the borrower. Use the mortality basis published by Finanstilsynet in their K2013 letter. You can take your age and legal gender as example. Use mortality risk. For the solution we will use: $z = 30$ in $Y = 2024$ and $G = R = 0$.

More concretely, carry out the following computations:

  (i) Estimate the loss distribution of the bank due to death.

 (ii) Compute the mean loss analytically and by Monte-Carlo. Compare the results.

(iii) Compute the solvency capital requirement, i.e. the 99.5%-quantile of the loss distribution.

Hint: the loss caused by death is naturally a distribution, whose values differ from person to person. The present value of the outstanding loan at time t is $V^+(t)$ from item (a). If τ denotes the death time of the borrower, then the relevant quantity to look at would be $v(\tau)V^+(\tau)\mathbb{I}_{\{\tau \leq T\}}$, which is the discounted

outstanding loan at the time of death where a loss happens only if death occurs before expiration of the loan.

Solution:

To generate a histogram we simply generate lives (in units of months) and look at compute the values $v(\tau)V^+(\tau)$. Next figure shows the result.

Taking the perspective of insurance: the cost would correspond to the present value of a death benefit corresponding to the prospective value of the loan at the time of death. Hence, a term insurance with benefit $V^+(t)$. This means

$$a_{*\dagger}(t) = V^+(t), \quad t \in [0, T).$$

The present value of such policy would be

$$V(0) = \int_{[0,\infty)} v(s) a_{*\dagger}(s) dN_{*\dagger}^Z(s) = \int_{[0,\infty)} v(s) V^+(s) \mathbb{I}_{[0,T]}(s) dN_{*\dagger}^Z(s).$$

Now, the integral has a unique jump of size 1 at $s = \tau$. Hence,

$$V(0) = v(\tau) V^+(\tau) \mathbb{I}_{[0,T]}(\tau).$$

The expectation (single premium) would be

$$\pi_0 = \mathbb{E}[V(0)] = \int_0^T v(s) V^+(s) f_\tau(s) ds,$$

where f_τ is the density function of τ (death time in months). We know the distribution of τ in years, which is given by Finanstilsynet (K2013),

$$\mathbb{P}[\tau/12 > t] = e^{-\int_0^t \mu_{Kol}(z+u, Y+u) du},$$

where z is the initial age and Y the starting year of the loan.

If s is a number of months then the distribution of the life time τ (in months) is

$$F_\tau(s) = 1 - \mathbb{P}[\tau > s] = 1 - e^{-\int_0^{s/12} \mu_{Kol}(z+u, Y+u) du}.$$

Hence, the density of τ is given by

$$f_\tau(s) = \frac{1}{12} e^{-\int_0^{s/12} \mu_{Kol}(z+u, Y+u) du} \mu_{Kol}(z + s/12, Y + s/12).$$

Finally,

$$\pi_0 = \mathbb{E}[V(0)] = \int_0^T v(s) V^+(s) f_\tau(s) ds,$$

which is exactly the formula for the *expected prospective value* at time $t = 0$ of a term insurance with death benefit equal to the prospective value $a_{*\dagger}(t) = V^+(t)$. See and compare with Theorem 5.1.2 or Corollary 5.1.3 from the lecture notes. In units of years the formula looks like

$$\pi_0 = \mathbb{E}[V(0)] = \int_0^{T/12} v(12s)V^+(12s)e^{-\int_0^s \mu_{Kol}(z+u, Y+u)du}\mu_{Kol}(z+s, Y+s)ds.$$
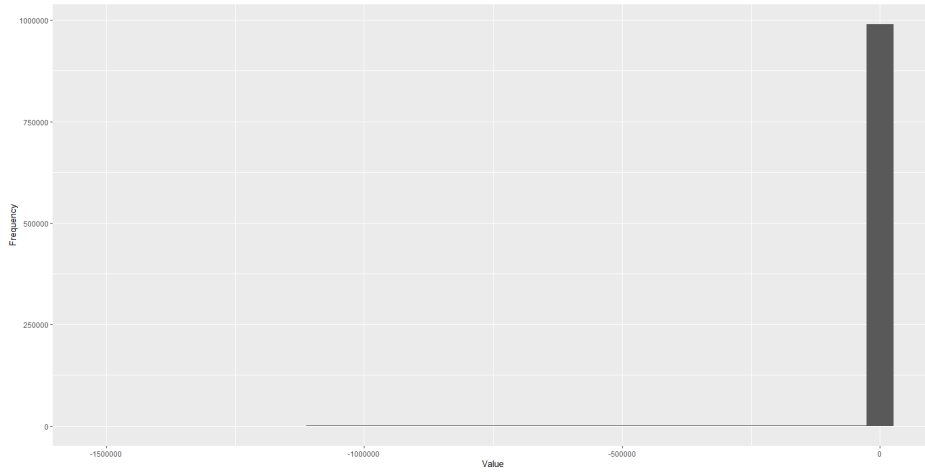
We obtained
$$\pi_0 = -5\,539.67\,.$$

Figure 6: Histogram of $V(0) = v(\tau)V^+(\tau)\mathbb{I}_{\{\tau \leq t\}}$, where τ is death time in months, based on $1\,000\,000$ simulations. Empirical mean $-5\,387.72$. Seed used:1. Execution time: $\sim$ 2h 8m.

The histogram shows very few deaths, since we are looking at a $z = 30$ year old borrower in 2024 for a period of 20 years. The probability of surviving from 30 to 50 is very high in Norway (according to the K13 model it is $p_{**}(0, 20) = 0.9892$), and even higher for females. You can try plotting an histogram for the loss distribution when the loan is granted to a, say 50 and 70 year old individual. You will see a different shape. We did it for the sake of fun.
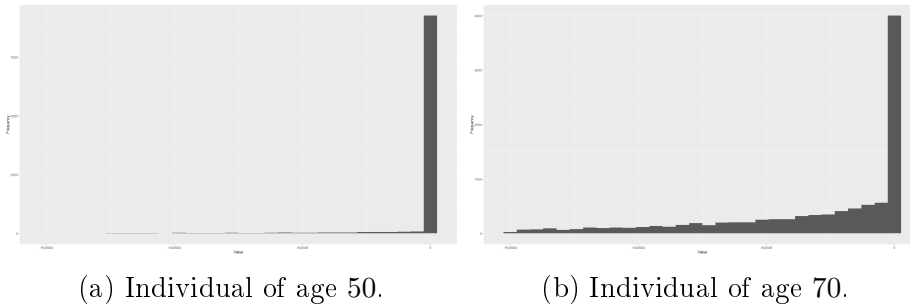
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(a) Individual of age 50.   (b) Individual of age 70.

Figure 7: Histogram of $V(0) = v(\tau)V^+(\tau)\mathbb{I}_{\{\tau \leq t\}}$ based on $10\,000$ simulations for ages $z = 50$ (left) and $z = 70$ (right). Seed used:1.

Back to our exercise, we computed the empirical mean for $n \in \{10^3, 10^4, 10^5, 10^6\}$ simulations, using seed 1. The results are collected in the next table.

Empirical quantities using Monte-Carlo				
nr. simulations	10^3	10^4	10^5	10^6
Mean	$-6\,268.40$	$-4\,514.89$	$-5\,479.8$	$-5\,387.72$
99.5%-solvency capital	$-331\,462.4$	$-355\,576.6$	$-456\,761.8$	$-429\,345.8$
Execution time	15s	1m 20s	$\sim$ 13m	$\sim$ 2h 8m

Table 2: Empirical premium computed by generating lives, computing loss and taking the mean. Seed used: 1. We see the importance of the accuracy in the number of simulations which comes at a computational cost.

The mean using the exact formula is

$$\pi_0 = \mathbb{E}[V(0)] = -5\,539.67 \text{ kr.}$$

We see that they are close as we increase the number of simulations, although it took at least $100\,000$ simulations to get close and quite some time for the empirical one!

```
library(ggplot2)
source("K2013.R")
source("probabilityK2013.R")
source("lifeK2013.R")
set.seed(1)

z <- 30 #age at year Y
Y <- 2024 #current year of choice
G <- 0 #gender
R <- 0 #risk
```

```
sim <- 100000 #number of lives
tau <- 12*(life.K13(z,sim)-z) #simulating lives (in months)
loss <- v(tau)*pV(tau)*(tau<T) #loss distribution
df <- data.frame(loss)
ggplot(df,
    aes(x=loss))+geom_histogram()+xlab("Value")+ylab("Frequency")

#Exact mean
f2 <- function(s) (1/12)*v(s)*pV(s)*p.K13(0,s/12)*my.kol(z+s/12 ,Y+
    s/12)
pi0 <- integrate(Vectorize(f2),0,T, subdivisions=500)$value
#Comparison
mean(loss)
pi0
#Empirical quantile
quantile(-loss,0.995)
```

(e) Compute the monthly premium from item (d), i.e. the deferred premium on
a monthly basis for that policy. Assume that premiums are paid monthly in
advance and the last payments occurs on the first day of the 239th month.

Solution:

The total cost of the loss due to demise of the borrower was computed in the
previous item, given by π_0. This is the quantity that the borrower should
pay the bank to hedge against the risk of losing the house in case of death.
The bank can offer the borrower to pay π_0 by means of monthly (or yearly)
payments. The idea is to «spread» the amount π_0 along the $T = 240$ months
of the contract. To do so, we create an «artificial» policy which consists of
payment of premiums on a monthly basis.

The policy function describing this setting is given by

$$a_*^{\text{Prem}}(t) = \begin{cases} -\pi, & t \in [0,1), \\ -2\pi, & t \in [1,2), \\ \dots, \\ k\pi, & t \in [k, k+1), \\ \dots, \\ \pi(T-1), & t \in [(T-1), \infty). \end{cases}$$

The policy cash flow is then given by

$$dC^{\text{Prem}}(s) = I_*^Z(s)da_*^{\text{Prem}}(s).$$

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Then, we need to compute the expected present value of this policy cash flow and match it with π_0 so that we can retrieve the monthly quantity π to be paid.

The expected present value (at $t = 0$) of this policy cash flow is given by

$$V_*(0, C^{\mathrm{Prem}}) = \int_{[0,\infty)} v(s)p_{**}(0, s)da_*^{\mathrm{Prem}}(s) = -\pi \sum_{k=1}^{T-1} v(k)p_{**}(0, k).$$

Hence, we need to impose that

$$V_*(0, C) + V_*(0, C^{\mathrm{Prem}}) = 0,$$

where C is the loan cash flow and C^{Prem} is the cash flow of premiums. Here, $\pi_0 = V_*(0, C)$ is the single premium. Hence, the equation for π that we obtain is

$$\pi_0 = \pi \sum_{k=1}^{T-1} v(k)p_{**}(0, k)$$

and finally,

$$\pi = \frac{\pi_0}{\sum_{k=1}^{T-1} v(k)p_{**}(0, k)}.$$

The value we obtain is:

$$\pi = 36.82 \text{ kr.}$$

This is a reasonable monthly premium for such a young borrower. Usually, the bank charge us more than that per month for administrative and other type of costs.

Note that we use a continuous time model, but impose that payments are made at discrete times, for this reason we obtain sums instead of integrals. If you model the payment of premiums on a continuous basis (as if one were paying daily, kind of) then you would have obtained integrals, like some of the examples in the lecture notes.

The R-code used was:

```
aux <- 0
for(k in 1:(T-1)){
  aux <- aux + v(k)*p.K13(0,k/12)
}
pi <- -pi0/aux
```

Problem 2: Disability insurance with retirement

We consider a multistate-state Markov model with three states in continuous time, see Figure 8. Z will be the Markov process with states in $\mathcal{Z} = \{0, 1, 2\}$ where 0 is the active/alive state, 1 is the disability state and 2 is the life ending state.
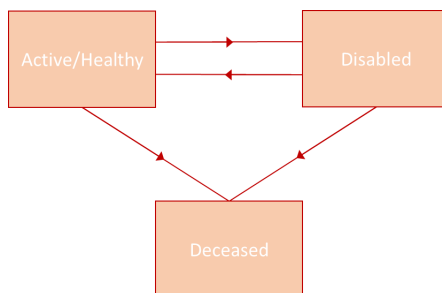

Figure 8: Disability model.

The transition rates that we will assume are contained in the following transition rate matrix

$$\Lambda(t) = \begin{pmatrix} \mu_{00}(t) & \mu_{01}(t) & \mu_{02}(t) \\ \mu_{10}(t) & \mu_{11}(t) & \mu_{12}(t) \\ \mu_{20}(t) & \mu_{21}(t) & \mu_{22}(t) \end{pmatrix}.$$

Biometric specifications: Assume the following transition rates

$$\mu_{01}(x) = 0.0004 + 10^{0.06x-5.46}, \qquad \mu_{02}(x) = 0.0005 + 10^{0.038x-4.12},$$
$$\mu_{10}(x) = 0.05, \qquad \mu_{12}(x) = \mu_{02}(x),$$

where x is age.

Policy specifications: Policyholder is a $z = 30$ year old individual with right to a disability pension of $D = 100\,000$ monetary units per annum as long as they are in the disability state from contract inception until retirement time $T_0 = 40$ years after inception and right to pension of $P = 300\,000$ monetary units per annum from retirement time T_0 to expiry time $T = 80$ as long as they are in the active/alive or disability state.

Economic/financial specification: We assume a technical constant interest rate of $r = 3\%$ per annum.

(a) Find the transition probabilities for this model by solving Kolmogorov's equation numerically.

   Then, make a program that simulates random lives, i.e. random paths of $t \mapsto Z(t, \omega)$ where ω is the life of an individual. An example of what a path should look like is given in Figure 9 below.
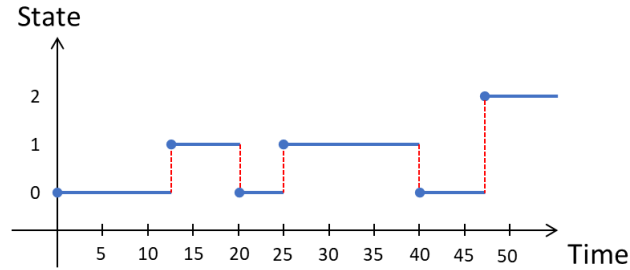
Figure 9: Example of a life performance of an individual.

To simulate lives from this model, you may use the following algorithm:

1. Take a partition of $[0, T]$ of size h and let $t_i = ih$, $i = 0, \ldots, n$ where $n = \frac{T}{h}$.

2. Set $Z(0) = 0$.

3. For $i = 1, \ldots, n$, generate a random observation, say j_i, from $\{0, 1, 2\}$ with the probabilities $\{p_{Z(t_{i-1}),0}(t_{i-1}, t_i), p_{Z(t_{i-1}),1}(t_{i-1}, t_i), p_{Z(t_{i-1}),2}(t_{i-1}, t_i)\}$. Set $Z(t_i) = j_i$.

After completing the algorithm you will have a vector of $n + 1$ states, one for each time point t_i, $i = 0, \ldots, n$ and where $Z(0) = 0$. Plot two, three or four random paths in one figure.

Solution:

To obtain the transition probabilities we need to solve Kolmogorov's equation which, in matrix form, is given by

$$\frac{d}{dt} P(s, t) = P(s, t) \Lambda(t), \quad t \geq s,$$

where $P(s, t)$ is the matrix with entries $p_{ij}(s, t)$ and Λ is the matrix with entries $\mu_{ij}(t)$. We used a classical Euler scheme with monthly steps $h = 1/12$, i.e.

$$P(s, t + h) \approx P(s, t) + h P(s, t) \Lambda(t) = P(s, t) \left(I + h \Lambda(t) \right),$$

where I stands for the identity matrix. Then, we start from $s = 30$ and, $P(30, 30) = I$ and compute $P(30, 30 + h)$ as

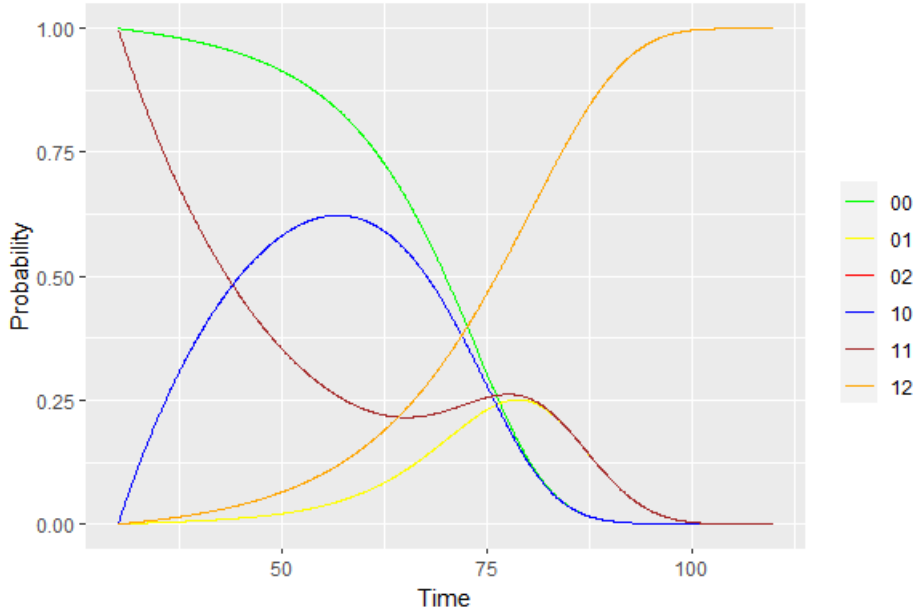$$P(30, 30 + h) \approx P(30, 30) \left(I + h \Lambda(30) \right)$$

and so on.

Figure 10: Transition probabilities $p_{ij}(30, s)$ for $s \in [30, 12]$ and $i, j \in \{0, 1, 2\}$. There is an overlap for p_{02} and p_{12} which are the red and orange ones, respectively. This is because $\mu_{02}(t) = \mu_{12}(t)$ in this exercise, otherwise they would not overlap.

To simulate paths, we note that at each step we need to sample from $\{0, 1, 2\}$ with vector of probabilities $(p_{i0}(kh, (k + 1)h), p_{i1}(kh, (k + 1)h), p_{i2}(kh, (k + 1)h))$ for $i \in \{0, 1, 2\}$ and $k = 0, \ldots, n$. These vectors correspond to the ith row of the transition probability matrix $P(kh, (k+1)h)$. Note that $(k+1)h - kh = h$ which is a small time step. Hence, we can use the following numerical approximation for the derivative

$$\frac{d}{dt} P(s, t) \approx \frac{P(s, t + h) - P(s, t)}{h}$$

with $s = t = kh$. Hence, we get that

$$P(kh, (k + 1)h) \approx P(kh, kh) \left(I + h\Lambda(kh) \right) = I + h\Lambda(kh).$$

The above is nothing else than Euler's method.

The algorithm we use is as follows. We fix an initial age $z = 30$ and a final age 110. Then, for each $t_k = z + kh$ for $k = 0, \ldots n$, where n is the number of points such that $z + nh = 110$, we compute

$$P(t_k, t_{k+1}) = P(z + kh, z + (k + 1)h) \approx I + h\Lambda(z + kh).$$

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Starting from $k = 0$, we have $Z(0) = 0$, then we pick the first row which corresponds to $(p_{00}(0, h), p_{01}(0, h), p_{02}(0, h))$ and sample from $\{0, 1, 2\}$ with the given probabilities. We set $Z(h)$ equal to the sampled value and proceed to retrieve $P(h, 2h)$ and pick the row corresponding to $Z(h)$ to generate the next value and so on. See the R code employed further below.

We generated 100 lives and selected the first four, displayed in the figure below.
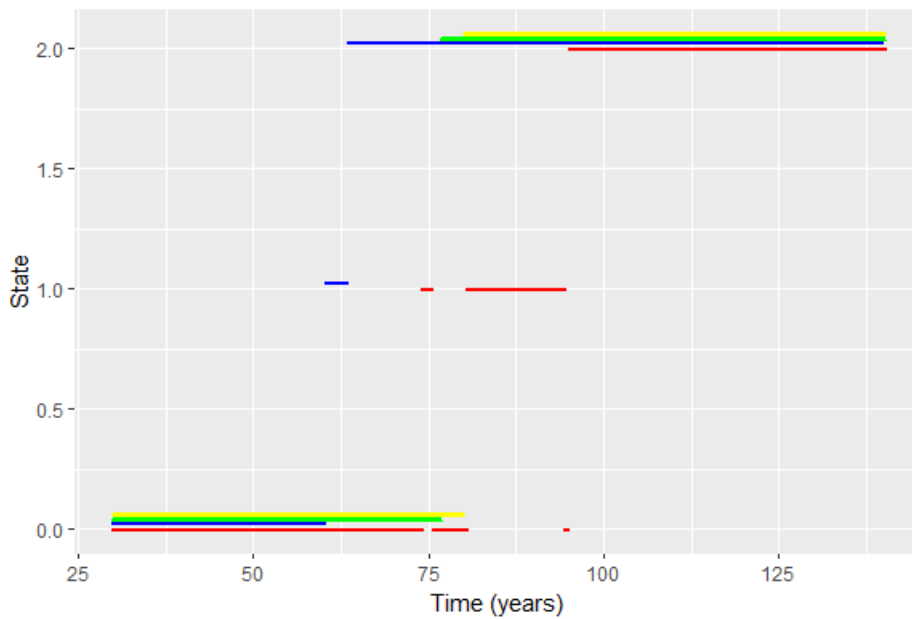

Figure 11: Example of four randomly generated life paths of an active 30 year old individual. Seed used: 1.

The figure shows the random performance of four individuals. We can see that the blue one was disabled for a short period of time before death. The red one had two periods of illness and the yellow and green ones had no periods of illness. As pointed out in the exercise, we will only consider subsidy for illness before retirement, i.e. before age 70, where pensions start.

We used the following code:

```
library(ggplot2)
library(latex2exp)

L <- function(u){
  mu01 <- 0.0004+10^(0.06*u-5.46)
  mu02 <- 0.0005+10^(0.038*u-4.12)
  mu00 <- -mu01-mu02
```

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```r
  mu10 <- 0.05
  mu12 <- 0.0005+10^(0.038*u-4.12)
  mu11 <- -mu10-mu12
  val <- matrix(c(mu00,mu01,mu02,mu10,mu11,mu12,0,0,0), nrow=3,
      byrow=TRUE)
  return(val)
}

field <- function(t,M){
  return(M%*%L(t))
}

Euler <- function(t0,P0,h,tn){
  if(t0==tn){ return(P0)}
  N<- ceiling((tn-t0)/h)
  D <- dim(P0)[1]
  #Initial condition at s
  P <- array( P0 , dim=c(D,D,N+1) )
  #First iteration
  P[,,1] <- P0 + h*field(t0,P0)


  for(n in 1:N){
    P[,,n+1] <- P[,,n]+h*field(t0+n*h,P[,,n])
  }
  return(P)
}

t0 <- 30
P0 <- diag(3)
tn <- 110
h <- 1/12

Euler_sol <- Euler(t0,P0,h,tn)

df <- data.frame(Time=seq(t0,tn,by=h), p00=Euler_sol[1,1,],
    p01=Euler_sol[1,2,], p02=Euler_sol[1,3,], p10=Euler_sol[2,1,],
    p11=Euler_sol[2,2,], p12=Euler_sol[2,3,])
ggplot(data = df, aes(x = Time)) +
  geom_line(aes(y = p00, colour = "00")) +
  geom_line(aes(y = p01, colour = "01")) +
  geom_line(aes(y = p02, colour = "02")) +
  geom_line(aes(y = p10, colour = "10")) +
```

```
    geom_line(aes(y = p11, colour = "11")) +
    geom_line(aes(y = p12, colour = "12")) +
    ylab("Probability")+
    scale_colour_manual("",
                        breaks = c("00", "01", "02", "10", "11", "12"),
                        values = c("green", "yellow", "red", "blue",
                                   "brown", "orange") )
h <- 1/12
N <- 110/h
sim <- 1000 #number of individuals
life <- matrix(rep(0,(N+1)*sim), nrow=N+1, byrow=TRUE)
z <- 30
#set.seed(1)

for(j in 1:sim){
  for(i in 1:N){
    P <- diag(3)+h*L(z+i*h) #starting from 30 yo individual
    p <- P[life[i,j]+1,]
    life[i+1,j] <- sample(c(0,1,2),1,p,replace=TRUE)
  }
}

time <- z+seq(0,N*h,by=h)
df <- data.frame(time,life[,1],life[,2],life[,3],life[,4])

ggplot(df)+
  geom_point(aes(x = time, y = life[,1]),color = "red", shape=0,
      size=0.1)+
  geom_point(aes(x = time, y = life[,2]+0.02),color = "blue",
      shape=1, size=0.1)+
  geom_point(aes(x = time, y = life[,3]+0.04),color = "green",
      shape=2, size=0.1)+
  geom_point(aes(x = time, y = life[,4]+0.06),color = "yellow",
      shape=3, size=0.1)+xlab("Time (years)")+ylab("State")
```

(b) Write down the policy functions of this insurance policy, the policy cash flow C described by the policy functions and the associated present, retrospective and prospective values of the policy cash flow.

Solution: The policy functions of interest are a_0, a_1, a_{02} and a_{12}. See Definition 4.3.4 for their definition.

The policy functions for sojourns a_0 and a_1 are given by

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$$a_0(t) = \begin{cases} 0, & t \in [0, T_0), \\ P(t - T_0), & t \in [T_0, T), \\ P(T - T_0), & t \in [T, \infty). \end{cases} \qquad a_1(t) = \begin{cases} Dt, & t \in [0, T_0), \\ DT_0 + P(t - T_0), & t \in [T_0, T) \\ DT_0 + P(T - T_0), & t \in [T, \infty). \end{cases}$$

The two policy functions above determine the entire policy benefits. If you wish to introduce a benefit due to death from any of the two living states (healthy and disabled) and at any time during the contract you can do so by including the policy functions for transitions a_{02} and a_{12}, which are given by

$$a_{02}(t) = \begin{cases} B, & t \in [0, T), \\ 0, & t \in [T, \infty). \end{cases} \qquad a_{12}(t) = \begin{cases} B, & t \in [0, T), \\ 0, & t \in [T, \infty). \end{cases}$$

However, the exercise did not ask for a death benefit, yet, so we just consider a_0 and a_1.

A general insurance cash flow in multistate modelling (see Definition 4.3.5) looks like

$$dC(s) = \sum_{i \in \mathcal{Z}} I_i^Z(s) da_i(s) + \sum_{\substack{i,j \in \mathcal{Z} \\ j \neq i}} a_{ij}(s) dN_{ij}^Z(s), \quad s \in [0, \infty).$$

Now that we have identified our policy functions we can find C, which is given by

$$dC(s) = I_0^Z(s) da_0(s) + I_1^Z(s) da_1(s).$$

Since the functions a_0 and a_1 are continuous and a.e. differentiable with derivatives $a_0'(s) = P\mathbb{I}_{(T_0,T)}(s)$ and $a_1'(s) = D\mathbb{I}_{(0,T_0)}(s) + P\mathbb{I}_{(T_0,T)}(s)$, respectively, we can simplify $dC(s)$ slightly as follows,

$$dC(s) = P\mathbb{I}_{(T_0,T)}(s) I_0^Z(s) ds + \left(D\mathbb{I}_{(0,T_0)}(s) + P\mathbb{I}_{(T_0,T)}(s) \right) I_1^Z(s) ds.$$

More compactly,

$$dC(s) = \left(D\mathbb{I}_{(0,T_0)}(s) I_1^Z(s) + P\mathbb{I}_{(T_0,T)}(s) \left(I_0^Z(s) + I_1^Z(s) \right) \right) ds.$$

The above cash flow says that a pension P is paid out (infinitesimally) during the retirement phase (T_0, T) both when being in state 0 or 1, while the disability pension D is paid out (infinitesimally) during the working phase $(0, T_0)$ only when being in state 1 (disabled).

The present, retrospective and prospective values are defined by

$$V(t,C) = \frac{1}{v(t)} \int_{[0,\infty)} v(s)dC(s),$$

$$V^-(t,C) = \frac{1}{v(t)} \int_{[0,t]} v(s)dC(s),$$

$$V^+(t,C) = \frac{1}{v(t)} \int_{(t,\infty)} v(s)dC(s),$$

where $v(t) = e^{-\int_0^t r(s)ds}$ is the discount factor. Hence, we have

$$V(t,C) = \frac{D}{v(t)} \int_0^{T_0} v(s)I_1^Z(s)ds + \frac{P}{v(t)} \int_{T_0}^T v(s)(I_0^Z(s) + I_1^Z(s))ds.$$

For $V^\pm(t,C)$ the integrals are similar, but the limits of the integrals need to be adjusted accordingly. We have,

$$V^-(t,C) = \frac{D}{v(t)} \int_0^{T_0 \wedge t} v(s)I_1^Z(s)ds + \frac{P}{v(t)} \int_{T_0 \wedge t}^{T \wedge t} v(s)(I_0^Z(s) + I_1^Z(s))ds$$

and

$$V^+(t,C) = \frac{D}{v(t)} \int_t^{T_0 \vee t} v(s)I_1^Z(s)ds + \frac{P}{v(t)} \int_{T_0 \vee t}^{T \vee t} v(s)(I_0^Z(s) + I_1^Z(s))ds.$$

(c) Plot the expected prospective values $V_0^+(t)$ and $V_1^+(t)$ for $t \in [0,T]$ and explain what these functions mean.

Solution:

The expected values $V_0^+(t)$ and $V_1^+(t)$ are obtained by taking conditional expectation $\mathbb{E}[\cdot|Z(t) = 0]$ and $\mathbb{E}[\cdot|Z(t) = 1]$, respectively of the (random) prospective value $V^+(t)$. That is

$$V_0^+(t) = \mathbb{E}[V^+(t)|Z(t) = 0], \quad V_1^+(t) = \mathbb{E}[V^+(t)|Z(t) = 1].$$

There are formulas for $V_0^+(t)$ and $V_1^+(t)$ that can be found in Theorem 5.1.2 from the lecture notes. The general formula is

$$V_i^+(t) = \frac{1}{v(t)} \sum_j \int_t^\infty v(s)p_{ij}(t,s)da_j(s) + \frac{1}{v(t)} \sum_{k \neq j} \int_t^\infty v(s)p_{ij}(t,s)\mu_{jk}(s)a_{jk}(s)ds.$$

Hence, in our case we have

$$V_0^+(t) = \frac{D}{v(t)} \int_t^{T_0 \vee t} v(s)p_{01}(t,s)ds + \frac{P}{v(t)} \int_{T_0 \vee t}^{T \vee t} v(s)(p_{00}(t,s) + p_{01}(t,s))ds$$

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and

$$V_1^+(t) = \frac{D}{v(t)} \int_t^{T_0 \vee t} v(s) p_{11}(t,s) ds + \frac{P}{v(t)} \int_{T_0 \vee t}^{T \vee t} v(s)(p_{10}(t,s) + p_{11}(t,s)) ds.$$

However, plotting these quantities numerically appears to be quite cumbersome. It can be done, but it requires that, for every time t, we need to solve Kolmogorov's equations to find $p_{ij}(t,s)$ for every $s \in [t,T]$ and then plug them into the integrals.

Thiele's equation allow us to compute V_0^+ and V_1^+ without needing p_{ij}. You may think of Thiele's equation as a combination of Kolmogorov's equations and numerical computation of the integrals in the expressions above.

Thiele's equation (see Theorem 5.4.3) in general is given by

$$\frac{d}{dt} V_i^+(t) = r(t) V_i^+(t) - a_i'(t) - \sum_{\substack{j \in \mathcal{Z} \\ j \neq i}} \mu_{ij}(t) \left(a_{ij}(t) + V_j^+(t) - V_i^+(t) \right),$$

where in our case we only have two relevant states, since $V_2^+(t) = 0$ for all t. Moreover, in our case we have and $a_{ij} = 0$ and $a_0'(t) = P\mathbb{I}_{(T_0,T)}(t)$ and $a_1'(t) = D\mathbb{I}_{(0,T_0)}(t)$. Hence, Thiele's equation for V_0^+ and $V_1^+(t)$ (which are intertwined) becomes

$$\frac{d}{dt} V_0^+(t) = r V_0^+(t) - P\mathbb{I}_{(T_0,T)}(t)) - \mu_{01}(z+t) \left(V_1^+(t) - V_0^+(t) \right) - \mu_{02}(z+t) \left(V_2^+(t) - V_0^+(t) \right),$$

$$\frac{d}{dt} V_1^+(t) = r V_0^+(t) - D\mathbb{I}_{(0,T_0)}(t) - \mu_{10}(z+t) \left(V_0^+(t) - V_1^+(t) \right) - \mu_{12}(z+t) \left(V_2^+(t) - V_1^+(t) \right),$$

with final condition $V_0^+(T) = V_1^+(T) = 0$. Recall that $V_2^+(t) = 0$ so,

$$\frac{d}{dt} V_0^+(t) = r V_0^+(t) - P\mathbb{I}_{(T_0,T)}(t)) - \mu_{01}(z+t) \left(V_1^+(t) - V_0^+(t) \right) + \mu_{02}(z+t) V_0^+(t),$$

$$\frac{d}{dt} V_1^+(t) = r V_0^+(t) - D\mathbb{I}_{(0,T_0)}(t) - \mu_{10}(z+t) \left(V_0^+(t) - V_1^+(t) \right) + \mu_{12}(z+t) V_1^+(t),$$

with final condition $V_0^+(T-) = V_1^+(T-) = 0$.

Obs: the transition probabilities $p_{ij}(t,s)$ and rates $\mu_{ij}(t)$ need to be age-adjusted. At time t the insured is $z+t$ years old, that is why we finally wrote $\mu_{ij}(z+t)$ in Thiele's equation. We did not write z before for notational ease.

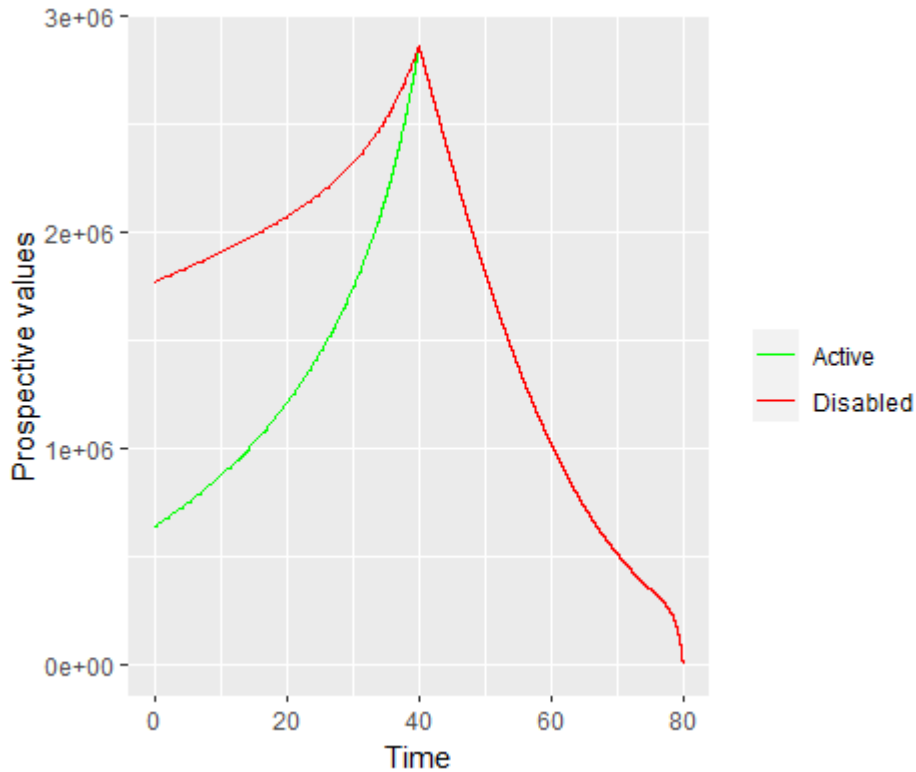We present the quantities $V_0^+(t)$ and $V_1^+(t)$ for each $t \in [0,T]$ in the next figure.

Figure 12: Expected prospective values, $V_0^+(t)$ and $V_1^+(t)$, in the two possible states: alive, disabled, at each time of the contract. We used a rather small step size $h = 0.0001$ for better accuracy.

The functions $t \mapsto V_i^+(t)$ for states $i = 0, 1$ are the *expected* cost of the remaining future contractual payments from t to end of contract (hence prospective), assuming that the insured is in state i at the time t of interest. Obviously, if the insured is in the disabled state, before retirement T_0, we need to expect more loss due to the disability pension. As time approaches retirement T_0 both expected values merge, since it is indifferent whether the insured is disabled or not after retirement. The pension P will be paid out anyway.

We know that the single premium is the initial value of the insurance, which is given by

$$\pi_0 = V_0(0, C) = \underbrace{\Delta a_0(0)}_{=0} + \underbrace{V_0^+(0, C)}_{\substack{\text{first value in} \\ \text{the green line}}} = 642\,019.9 \text{ kr.}$$

We used the following R-code as a continuation to the previous one. It make take some time to run, since $h = 0.0001$ here.

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```
#Policy
#k: time, j:life
h <- 0.0001
r <- 0.03
z <- 30 #age
T0 <- 40 #retirement (30+40=70)
T <- 80 #ending contract age (30+80=110)
P <- 300000 #yearly pension
D <- 100000 #yearly disability benefit
v <- function(s) exp(-r*s)
n <- T/h

#Expected prospective values via Thiele's equation
time <- seq(0,T,by=h)
nt <- length(time)-1
V0 <- V1 <- rep(0,nt+1)
da0 <- function(t){
  return(P*1*(T0<t)*(t<T))
}
da1 <- function(t){
  return(D*1*(0<t)*(t<T0)+P*1*(T0<t)*(t<T))
}

for(k in nt:1){
  V0[k] <- V0[k+1] - h*(r*V0[k+1] - da0(k*h) -
      L(z+k*h)[1,2]*(V1[k+1]-V0[k+1])+L(z+k*h)[1,3]*V0[k+1])
  V1[k] <- V1[k+1] - h*(r*V1[k+1] - da1(k*h) -
      L(z+k*h)[2,1]*(V0[k+1]-V1[k+1])+L(z+k*h)[2,3]*V1[k+1])
}

df.pV <- data.frame(time,V0,V1)
ggplot(df.pV, aes(x=time))+geom_line(aes(y = V0, colour = "Active"))+
    geom_line(aes(y = V1, colour =
    "Disabled"))+xlab("Time")+ylab("Prospective values")+
  scale_colour_manual("",
                      breaks = c("Active", "Disabled"),
                      values = c("green", "red") )
```

(d) Simulate a histogram of $V(0, C)$ using the algorithm in (a). Compute the empirical mean of $V(0, C)$ and the single premium using the exact formula from the lecture notes. Compare and discuss. Compute the yearly premiums of this insurance assuming that the insured pays premiums during the working

period only if they are in the active state and plot the function $t \mapsto V_*^+(t, C) + V_*^+(t, C^\pi)$, where C^π is the cash flow associated to the payment of premiums.

Solution:

At $t = 0$ the initial value of the insurance is given by the random variable

$$V(0, C) = D \int_0^{T_0} v(s) I_1^Z(s) ds + P \int_{T_0}^T v(s)(I_0^Z(s) + I_1^Z(s)) ds.$$

The distribution $V(0, C)$ consists of discounted benefits due to retirement (pension), illness and either a death benefit from state 0 or from state 1.

We simulate an histogram by using the outcomes generated from item (a) and applying the formula $V(0, C)$ above, that is, associating to each life performance its insurance cost. We obtain the following figure.
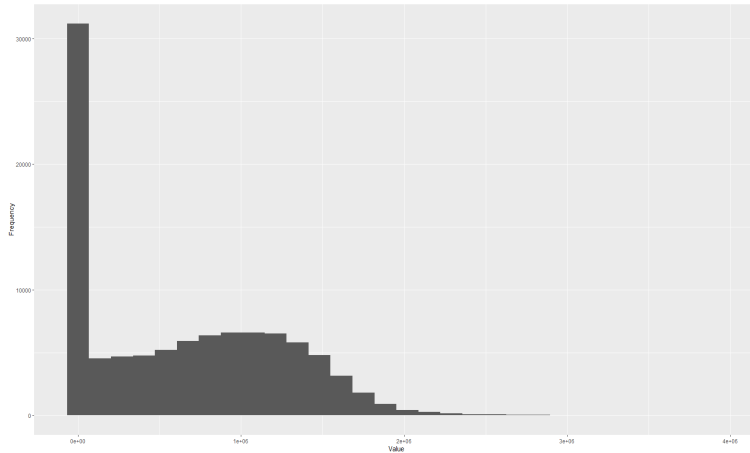

Figure 13: Histogram of $V(0)$ for $100\,000$ simulations. Seed used: 1. The empirical mean obtained is: $637\,706.7$ kr. Execution time: ~ 46 minutes.

We see that a significant part of the pool of lives costs 0 and some others cost some money, due to either disability, pension or both. It may seem strange that so many cost 0 (or around 0) since this actually means that they were not disabled and then died before retirement, but taking a closer look to the survival probability $p_{00}(z, z + T_0)$ we see that $p_{00}(z, z + T_0) \approx 0.5$ for the given (artificial) model, so it is not so surprising after all. If we want to make the model more realistic, we should tweak the values for μ_{ij} slightly.

The exact initial value of the policy is given by

$$\pi_0 = V_0^+(0) = D \int_0^{T_0} v(s) p_{01}(0, s) ds + P \int_{T_0}^T v(s) \left(p_{00}(0, s) + p_{01}(0, s) \right) ds.$$

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We computed it already using Thiele's equation, $\pi_0 = 642\,019.9$ kr.

We see that the theoretical premium (using Thiele) is «fairly» close to the empirical one having used $1\,000$ simulated lives, which took us some time. Recall that the empirical value of $637\,706.7$ is for the seed number 1, if we try other seeds we can obtain closer values to the exact one or even further away.

Now, we compute the yearly premiums π to be paid as long as the insured is in state 0. To do so, we keep the total cost of the insurance which is $\pi_0 = 642\,019.9$ and we create a «new» cash flow which pays 1 monetary unit a year only in state 0, i.e.

$$a_0^{\pi=1}(t) = \begin{cases} -t, & t \in [0, T_0), \\ -T_0, & t \in [T_0, \infty). \end{cases},$$

which essentially means that if the insured is in state 0 *all* the time and pay 1 unit per year, they will have paid T_0 during the working time which lasts for T_0 years. Since premiums are negative, hence the sign.

Now, we know that the present value of yearly π is given by

$$V_0^+(0, C^\pi) = \pi \int_0^{T_0} v(s) p_{00}(0, s) ds$$

and by the equivalence principle we have

$$0 = \underbrace{V(0, C)}_{=\pi_0} + V(0, C^\pi),$$

which implies
$$\pi = -\frac{\pi_0}{\int_0^{T_0} v(s) p_{00}(0, s) ds} \approx 30\,610.15 \text{ kr.}$$

```
#Initial (random) cost (histogram)
pV <- rep(0,sim)
for(j in 1:sim){
  aux <- 0
  for(k in 0:N){
    aux <- aux + D*h*v(k*h)*1*(life[k+1,j]==1)*(k*h>=0)*(k*h<T0) +
        P*h*v(k*h)*(1-1*(life[k+1,j]==2))*(k*h>=T0)*(k*h<T)
  }
  pV[j] <- aux
}

df <- data.frame(pV)
```

```
ggplot(df, aes(x=pV))+geom_histogram()+xlab("Value")+ylab("Frequency")

#Empirical premium vs exact premium
mean(pV)
pi0 <- V0[1]

#Deferred premium
aux1 <- 0
for(k in 1:(T0/h)){
  aux1 <- aux1 + h*v((k-1)*h)*Euler_sol[1,1,k]
}
pi <- pi0/aux1
```

(e) Consider now the case where a death benefit is paid out corresponding to the expected prospective value of the policy at death time (refund guarantee). Recompute the single premium of this policy when adding such benefit to the contract. Comment.

Solution:

This exercise is easy if we use Thiele's equation. Recall the general Thiele's equation (see Theorem 5.4.3)

$$\frac{d}{dt}V_i^+(t) = r(t)V_i^+(t) - a_i'(t) - \sum_{\substack{j \in \mathcal{Z} \\ j \neq i}} \mu_{ij}(t)\left(a_{ij}(t) + V_j^+(t) - V_i^+(t)\right),$$

Then in our case, if we keep policy functions a_{02} and a_{12} corresponding to death benefits we have

$$\frac{d}{dt}V_0^+(t) = rV_0^+(t) - P\mathbb{I}_{(T_0,T)}(t)) - \mu_{01}(z+t)\left(V_1^+(t) - V_0^+(t)\right) - \mu_{02}(z+t)\left(a_{02}(t) - V_0^+(t)\right),$$

$$\frac{d}{dt}V_1^+(t) = rV_0^+(t) - D\mathbb{I}_{(0,T_0)}(t) - \mu_{10}(z+t)\left(V_0^+(t) - V_1^+(t)\right) - \mu_{12}(z+t)\left(a_{12}(t) - V_1^+(t)\right).$$

Choosing $a_{02}(t) = V_0^+(t)$ and $a_{12}(t) = V_1^+(t)$ the prospective value itself as the exercise suggests, kills the contribution of the terms multiplied by μ_{02} and μ_{12}, respectively, giving thus rise to a much simpler Thiele's equation. Namely,

$$\frac{d}{dt}V_0^+(t) = rV_0^+(t) - P\mathbb{I}_{(T_0,T)}(t)) - \mu_{01}(z+t)\left(V_1^+(t) - V_0^+(t)\right),$$

$$\frac{d}{dt}V_1^+(t) = rV_0^+(t) - D\mathbb{I}_{(0,T_0)}(t) - \mu_{10}(z+t)\left(V_0^+(t) - V_1^+(t)\right),$$

with final condition $V_0^+(T-) = V_1^+(T-) = 0$.

Solving the above equation is a matter of using the same program as in item (c)
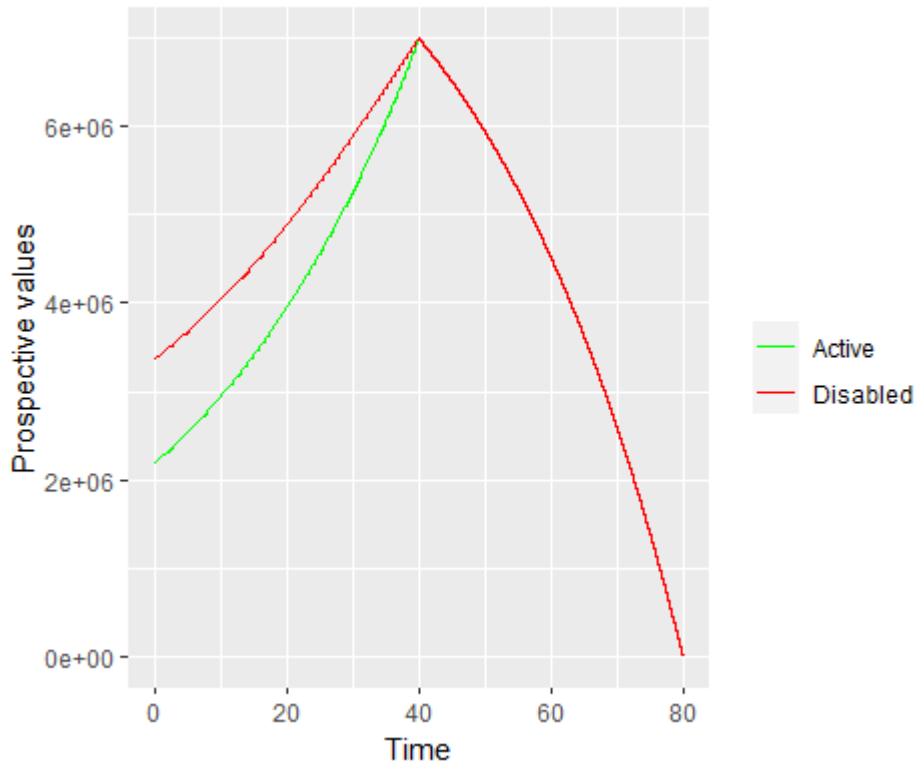removing the last term of each equation. This gives the following result:

Figure 14: Expected prospective values, $V_0^+(t)$ and $V_1^+(t)$, in the two possible states:
alive, disabled, at each time of the contract considering refund guarantee, in case of
death, corresponding to the prospective value in the last state. We used a rather
small step size $h = 0.0001$ for better accuracy.

The initial cost of this insurance (with death benefit) is:

$$\pi_0 = V_0(0, C) = \underbrace{\Delta a_0(0)}_{=0} + \underbrace{V_0^+(0, C)}_{\substack{\text{first value in} \\ \text{the green line}}} = 2\,186\,522 \text{ kr.}$$

It is much higher since we are always offering a death benefit. The mathematical
reason why it is higher can be seen by looking at Thiele's equation. In the equation
without death benefit, a positive term $\mu_{i2}(t)V_i^+(t)$ appears in equation $i = 0, 1$, thus
reducing the value of V_i^+. By removing this term, we increase the value of V_i^+.