STK4500: Life Insurance and Finance
Home exam - Spring 2020

This exam consists of 2 exercises with a total of 9 items. All items are worth 1 point, except for item (d) in Exercise 2 which is worth 2 points. The deadline is **Thursday 18th of June at 9:00 am**. Make sure to be precise when stating mathematical results and formulae. Prove the results you use, or refer to the lecture notes or the book. LaTeX or any other text editor is preferred, but if you do the exam by hand, please write clearly and avoid crossing out. Attach the code of the programs you use in an appendix. Good luck!

Finanstilsynet (The Financial Supervisory Authority of Norway) is a Norwegian government agency responsible for supervision of financial companies within Norway based on law and regulations from Stortinget, the Norwegian Ministry of Finance and international accounting standards. It is located in Oslo and is under the supervision of the Ministry of Finance.

Life insurance companies and pension funds make use of mortality rates suggested by Finanstilsynet in a letter from 8th of March, 2013, as a benchmark. You can find this letter by going to `finanstilsynet.no` and writing "K2013" in the search engine. The document is called "Nytt dødelighetsgrunnlag i kollektiv pensjonsforsikring". Or directly from this link:

https://www.finanstilsynet.no/contentassets/60b4b0182be14b69a0dd1a2a6ec019b2/nytt-doedelighetsgrunnlag-i-kollektiv-pensjonsforsikring-k-2013.pdf

On page 4 of this document, you see a model for the mortality in Norway based on data from 2013, where they have added an exponential trend due to longevity risk. They consider the following function

$$\mu_{Kol}(x,t) = \mu_{Kol}(x,2013)\left(1+\frac{w(x)}{100}\right)^{t-2013}, \quad t \in \{2013, 2014, 2015, \dots\}. \tag{0.1}$$

Here, $\mu_{Kol}(x,2013)$ is the mortality for a life of age x in 2013, whereas $\mu_{Kol}(x,t)$ is the mortality for a life aged x in the calendar year t (t being at minimum 2013). Further, we call $w(x)$ the mortality decrease, given by

$$\begin{aligned} w(x) &= \min\{2.671548 - 0.172480x + 0.001485x^2, 0\} \quad \text{for men,}\\ w(x) &= \min\{1.287968 - 0.101090x + 0.000814x^2, 0\} \quad \text{for women.} \end{aligned} \tag{0.2}$$

The mortality for longevity risk is given by the following formulas

$$\begin{aligned} 1000\mu_{Kol}(x,2013) &= 0.189948 + 0.003564 \cdot 10^{0.051x} \quad \text{for men,}\\ 1000\mu_{Kol}(x,2013) &= 0.067109 + 0.002446 \cdot 10^{0.051x} \quad \text{for women.} \end{aligned} \tag{0.3}$$

The mortality for mortality risk is given by the following formulas

$$\begin{aligned} 1000\mu_{Kol}(x,2013) &= 0.241752 + 0.004536 \cdot 10^{0.051x} \quad \text{for men,}\\ 1000\mu_{Kol}(x,2013) &= 0.085411 + 0.003114 \cdot 10^{0.051x} \quad \text{for women.} \end{aligned} \tag{0.4}$$

Exercise 1 (Norwegian orphan's pension)

For this exercise, consider the mortality proposed by Finanstilsynet given in (0.1) by choosing $t = 2020$ with weights (0.2) and (0.4).

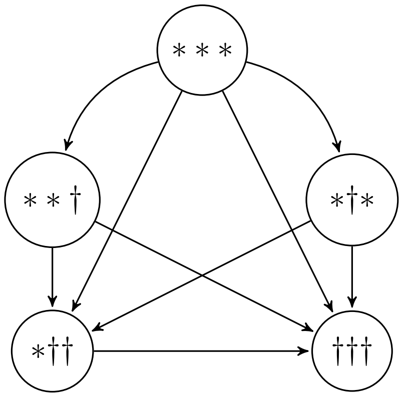
A couple consisting of two Norwegian men of ages 35 and 40 have a newborn male child and decide to write an insurance to protect the child against an early death of one or both fathers. We will model the states of this family using a regular Markov chain $X = \{X_t, t \geqslant 0\}$ with state space

$$S = \{(i,j,k), i,j,k \in \{*, \dagger\}\},$$

where i stands for the state of the child, j of the first parent and k the second parent. For instance, $(*,*,\dagger)$ means that the child and first parent are alive while the second parent is deceased. We assume that these three lives are independent of each other.

The policy pays a monthly pension of NOK 7 500 on the first day of each month, immediately after one of the parents dies and NOK 15 000 after the death of the second parent or simultaneous death of both of them. Monthly premiums P are paid as long as all three members of this family are alive and the contract ends in one of the following situations: the child dies or, the child turns 25 years.

The following diagram may be of help to see the possible transitions



Observe that transitions of the type $(ijk) \rightsquigarrow (\dagger jk)$, $i,j,k \in \{*, \dagger\}$ are omitted from the diagram since, in such case, the contract ends without benefits. Also, it is worth noting that in a continuous time setting, transitions from $(***)$ to $(*\dagger\dagger)$ are not possible (up to a null measure set) without passing through, e.g. $(**\dagger)$ etc. You can see this by looking at the definition of transition rates. This issue is ruled out by the fact that we consider a discrete time setting.

Here, it is natural to use the mortalities given by the formula for *death risk* in contrast to *survival risk* as the risk here is the early death of the parents. We take a constant annual technical interest rate of $\delta = 3\%$.

We will consider a discrete time model and denote by $a^{Pre}_{(ijk)}$, $i,j,k \in \{*, \dagger\}$ the pension payment which is due at time t, given that the insured is at time t in i. Notice that the capital benefits $a_{(ijk)(i'j'k')}$, $i,j,k,i',j',k' \in \{*, \dagger\}$ of this insurance are all zero.

Let T denote the maturity of this contract, i.e. $T = 25$ years. and by $V_i^+(t, A)$, $t \in [0,T]$ the value of this policy at each t given that $X_t = i$, $i \in S$, for example, $V^+_{*\dagger\dagger}(t, A)$ is the value of the contract at time $t \in [0,T]$ given that both parents have passed away. Or $V^+_{*\dagger *}(0, A)$ is the value of this contract when entering the contract, given that one parent is already missing. Obviously, $V^+_{\dagger\dagger\dagger}(t, A) = 0$ for all $t \geqslant 0$.

(a) Compute the present value of this policy (no premiums accounted) for the four possible cases: both parents are alive, the youngest parent is deceased, the oldest parent is deceased and, both parents are deceased. Display the results in a table and a figure (show only the yearly values). Comment.

(b) Compute the monthly premiums for this insurance. Compute the present value of the premiums and the mathematical reserves for this policy. Display the results in a figure. Give an interpretation and explanation of the reserves you have obtained.

(c) To make this insurance a bit more attractive to the customers, the company waives the premiums when the child turns 16, but the policy is still in force until he turns 25. Recompute the monthly premiums and reserves. Comment.

The Solvency II Directive (2009/138/EC) is a Directive in European Union law that codifies and harmonises the EU insurance regulation. Primarily this concerns the amount of capital that EU insurance companies must hold to reduce the risk of insolvency.

Following an EU Parliament vote on the Omnibus II Directive on 11 March 2014, Solvency II came into effect on 1 January 2016. This date had been previously pushed back many times.

You can take a look at it by following the link `https://eur-lex.europa.eu/eli/reg_del/2015/35/oj` and selecting the language that suits you, if not English.

One of the major risks a life-insurance faces is *interest rate risk* which arises in the variations of interest rates in the term structure or yield curve. Recall that in the formulas for reserves we have factors of the form

$$\frac{v(\tau)}{v(t)} = e^{-\int_t^\tau r_s ds} = e^{-(\tau-t)R(t,\tau)}, \quad 0 \leqslant t \leqslant \tau,$$

which correspond to the price of a zero-coupon bond at time t with maturities $\tau \geqslant t$. The function $\tau \mapsto R(t, \tau)$ is the so-called term structure or yield curve of interest rates. In our case, we simply take it flat, i.e. $R(t, \tau) = r = 3\%$.

The natural question arises on how different scenarios for interest rate term structure affect the reserves, and hence, the capital requirements of the company. Solvency II provides two different stress scenarios for this. Article 166 provides a stress scenario of an increase of the term structure of interest rates and, on the contrary, Article 167 provides a stress scenario of decrease. In Figure 1 you can see the results of stressing $r = 3\%$ upwards and downwards respectively using the values from Solvency II.

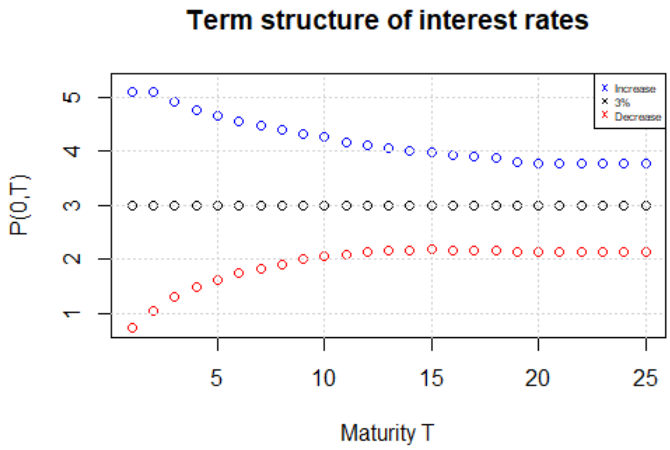


FIGURE 1

The figure above shows the function $\tau \mapsto R(0,\tau)$ in the three cases: inflating the forward rates, taking them flat and deflating them. Hence, you will need to adapt the factor $v(t)/v(\tau)$ in terms of $R(0,t)$ and $R(0,\tau)$ and use the values in the figure in order to compute the reserves. More concretely,

$$\frac{v(\tau)}{v(t)} = e^{-\tau R(0,\tau)+tR(0,t)},$$

where the $s \mapsto R(0,s)$ are given in Figure 1. Note that these rates are yearly. When applying them monthly you may just take the 12th root of the discount factors if you wish.

(d) Recompute the reserves inflating or deflating the interest rates according to Articles 166 and 167 from Solvency II and plot the reserves in these two cases. Report the differences.

(e) Take a look at Article 137. Compute the capital requirement for mortality risk as described in Article 137, i.e. the difference in reserves when mortality is increased by 15%.

Exercise 2 (Competing risks)

A model for competing risks in the context of insurance is used to distinguish between different causes of death. Competing risks methodology is being increasingly applied to cause of death data as a way of obtaining "real world" probabilities of death broken down by specific causes. This kind of information could be vital not only for informing patients of the risks they face in certain situations, but also for making decisions about which treatment regime to assign a patient, how best to allocate health resources and for understanding the longer term outcomes of chronic conditions.

In this exercise we will consider insurance policies related to the cause of death. To this end, we need statistics of this in order to obtain transition rates. We consider data from SSB (Statistics Norway). More specifically, data from 2012 on cases of death by age and gender. You may take a look at the statistics following these links https://www.ssb.no/statbank/table/08880 and https://www.ssb.no/statbank/table/07459. There, you can select what

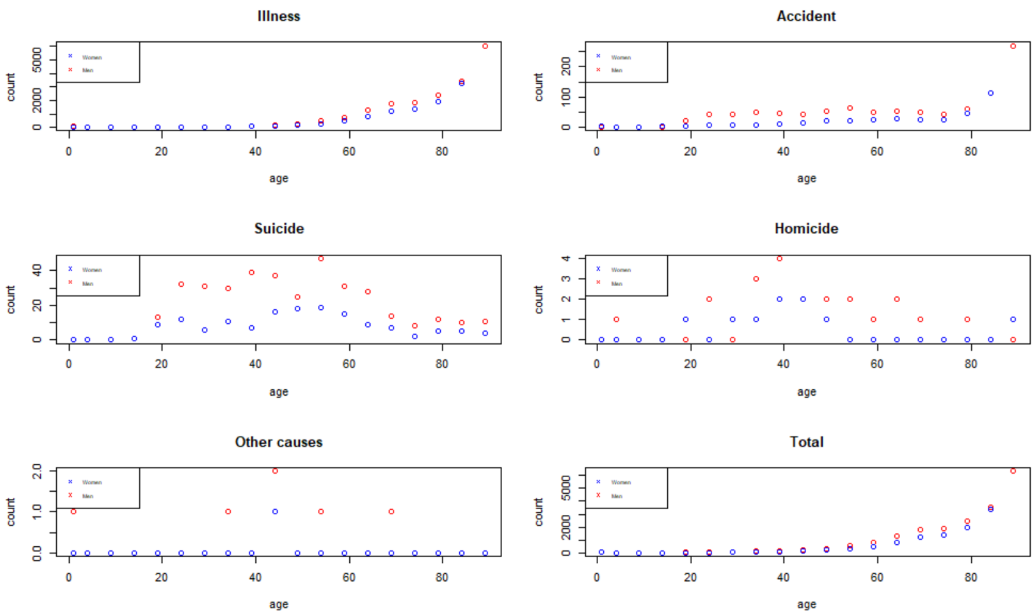


FIGURE 2: Death cause counts among men and women in Norway in 2012

data you wish to analyse. We have downloaded data for men and women, sorted by age and categorised by *illness*, *accident*, *suicide*, *homicide* and *other causes*, as well as data on the population by age and gender in 2012. We have arranged the data for you. The data consists of the rates by gender, age and death cause. In Figure 2 you can see a display of the frequencies from 2012.

As you can see in Figure 2, some patterns are clear and some are not so clear. We may agree that accidents and illnesses are the most common causes and that their risk increases by age. Hence, we can model these by using suitable exponential functions. On the contrary, suicide, homicide and other causes have somewhat irregular patterns and are more concentrated in certain ages. Consider an independent regular Markov chain $X = \{X_t, t \geqslant 0\}$ with state space $S = \{*, \dagger_1, \dots, \dagger_n\}$ where $*$ stands for "alive" and $\dagger_i$, $i = 1, \dots, n$ are different excluding causes of death. Here, $\dagger_1$ illness, $\dagger_2$ accident, $\dagger_3$ suicide, $\dagger_4$ homicide and $\dagger_5$ other causes. Denote by $\mu_{ij}(t)$, $t \geqslant 0$, $i, j \in S$, the transition rates of X. Estimating μ_{ij} is a very demanding task, and there are plenty of methods. To make the exercise less tedious we will just use empirical quantities. That is, we take as function $t \mapsto \mu_{ij}(t)$ the piecewise linear function passing through all observed rates. This implies that when you integrate these functions in order to get transition probabilities, you will just be applying the trapezoidal rule for numeric integration.

Observe that $\mu(t) \triangleq \sum_{i=1}^{5} \mu_{*\dagger_i}(t)$ corresponds to the overall mortality rate.

(a) For a regular Markov chain X with state space S, denote by $p_{ij}(s,t)$, $i, j \in S$ its transition probabilities, i.e. the probability that $X_t = j$ given that $X_s = i$, $s < t$. Then prove that,

$$p_{**}(s,t) = \exp\left(-\sum_{i=1}^{n} \int_s^t \mu_{*\dagger_i}(u)du\right), \quad s < t$$

and that

$$p_{*\dagger_i}(s,t) = \int_s^t p_{**}(s,u)\mu_{*\dagger_i}(u)du, \quad s < t, \quad i = 1, \dots, n.$$

Explain how you would estimate the transition probabilities using the data and display some graphs for each gender.

We consider a (continuous time) policy of $T > 0$ years with a stochastic technical interest rate $r = \{r_t, t \in [0, T]\}$ given by the Vasicek model, i.e.

$$dr_t = a(b - r_t)dt + \sigma dW_t, \quad r_0 > 0, \quad t \in [0, T],$$

where $a, b, \sigma \in \mathbb{R}$, $\sigma > 0$ and $W = \{W_t, t \in [0, T]\}$ is a standard Brownian motion.

Denote the generalised pension payments by a_i, $i \in S$ and the generalised capital benefits by a_{ij}, $i, j \in S$, $i \neq j$.

Consider an x-year-old Norwegian woman. The policy pays the sum E only if she is alive by time T and it pays the amount B_i if she dies of cause $\dagger_i$, $i = 1 \ldots, n$ before the policy expires. The contract is in force during the time span $[0, T]$. Let $X = \{X_t, t \geqslant 0\}$ denote the Markov chain of states of this client during the contract, i.e. $S = \{*, \dagger_i, i = 1, \ldots, n\}$.

(b) Assume $x = 50$, $T = 20$ and

$$\begin{aligned} E &= \text{NOK } 500\,000 \\ B_1 &= \text{NOK } 1\,000\,000 \\ B_2 &= \text{NOK } 2\,000\,000 \\ B_3 &= \text{NOK } 100\,000 \\ B_4 &= \text{NOK } 2\,000\,000 \\ B_5 &= \text{NOK } 200\,000 \end{aligned}$$

Further, consider the parameters $r_0 = 3\%$, $a = 0.5$, $b = 4\%$ and $\sigma = 5\%$ for the Vasicek model for the interest rate r.

Compute the present value of this insurance at time $t \in \{0, 1, 2, \ldots, T\}$ given that $X_t = *$. You will realise that the reserves in this case are stochastic (except from when $t = 0$) because of the stochastic interest rate. In order to provide useful statistics, display the mean reserves, standard deviations and quantiles or histograms.

(c) Compute the yearly premiums P of this insurance, their present value and the mathematical reserves for each time $t \in \{0, 1, 2, \ldots, T\}$.

(d) We will carry out a scenario test in order to estimate the distributions of reserves. Imagine that you sell 1 000 of such identical policies. Compute the reserves for each outcome ω_i, $i = 1, \ldots, 1\,000$ at the beginning of the contract. Recall that here, each outcome consists of a life performance of a 50-year-old Norwegian woman and an interest rate curve given by the Vasicek model (and that they are assumed to be independent processes). Nevertheless, in order to make the exercise simpler, we wil just take the expectation of the discount factor and only focus on generating life paths. Prove that the time at which a transition from $*$ to $\dagger_i$, $i = 1, 2, 3, 4, 5$ happens is a stopping time with respect to the filtration generated by both X and r and that it is furthermore a random variable (i.e. in addition $\mathbb{P}$-a.s. finite).

Display the results and comment. Display the empirical distribution of V_0^+. Report the value α such that

$$\mathbb{P}\left[\omega \in \Omega : V_0^+(\omega) \geqslant \alpha\right] = 0.05$$

Note: If you have time, you may try 10 000 paths instead in order to get more accurate results, the running time may take half an hour or so.

GOOD LUCK!