1 Financial derivatives

1.1 Introduction

Derivatives are contracts derived from primary assets. With *financial* derivatives those are equity, bonds, interest rates or currency. The idea is to pass some of the risk for an adverse development on to somebody else. Savings with guaranteed return is an example. A financial institution will then reimburse customers when earnings fall below a certain floor. Companies also seek protection themselves. If it is inconvenient to carry the entire burden of the stock market going down or interest rates dropping, it may pay another company to take over. The similarity to insurance and re-insurance is striking. What is different is the detailed mathematical models and above all a possibility of **hedging** financial risk that with insurance simply wasn't there.

To indicate why such considerations must influence the valuation of derivative contracts in a free market system consider an institution responsible for a financial option that will result in a pay-off when the stock market goes down. Contracts of that kind was called puts in Section 3.7. The risk of the option seller was

$$X = \max(r_g - R, 0)S_0$$

where S_0 is the value of the orginal shares and R the return of such investments. There is a pay-off if $R < r_g$. Now, introduce a second operation where an amount ΔS_0 of the underlying equity is sold short. Payment is then received today for shares rebought and delivered later. The net earning in this is $-R\Delta S_0$, gaining if the market has fallen, losing otherwise. When the option and the short position are added, the balance sheet becomes

$$\underbrace{-\max(r_g - R, 0)S_0}_{\textit{put option}} \quad - \quad \underbrace{R\Delta S_0}_{\textit{short selling}},$$

and as R changes the two contributions move in the opposite direction. The second operation **hedges** the first.

Risk is lowered if Δ is selected judisciously. In a well functioning market that must influence the price an option seller is able to charge. Indeed, it will emerge in Section 14.4 that option contracts turn into veritable money machines for the sellers if the prices are at the ordinary actuarial $E(X)$. But *how much* smaller than $E(X)$ should they be? That is a riddle with a neat solution if we are allowed to run *multiple* risk-reducing operations and fine-tune many times the stock ΔS_0 held on the side. The argument is subtle and works equally well with currency. Interest rate derivatives are different since the underlying asset is missing. We can't buy interest! What we *can* do, is to reduce risk through assets that *correlate* with the derivative, the stronger the better, and the idea remains the same. Prices must adjust to reflect the risk *after* the hedging has been taken into account. Interest rates derivatives are among the most important risk-reducing tools of all. It could be borrowers protecting themselves from *high* interest rates or the opposite, insurance companies with gurantees towards its customers fearing *low* rates. Names are colourful ones like swaps, caps, and floors. Interest rate derivatives are discussed in Sections 14.6 and 14.7.

This chapter is an informal introduction to the subject, sufficient to get the main ideas, and it will enable us to incorporate risk-reducing strategies in the ensuing Chapter 15. The formalism of stochastic analysis is *not* used. An elementary text supplementing the material in that direction is Neftci (2003).

1.2 Arbitrage and risk neutrality

Introduction
American economists Fischer Black, Robert Merton and Myron Scholes started in 1973 an avalanche of research into the consequences of assuming financial markets to be free of **arbitrage**; see Black and Scholes (1973) or Merton (1973). Let r be the risk-free rate of interest, in much of the chapter continously compounded so that an ordinary bank account over a period of length T grows by the factor $\exp(rT)$. No arbitrage means that it is impossible to set up a financial portfolio for which we can be certain that a future value $\mathcal{X}$ exceeds the initial value $\mathcal{X}_0$ added risk-free interest. In other words, things can *not* be arranged so that *with certainty*

$$\mathcal{X} \geq e^{rT}\mathcal{X}_0 \qquad \text{and sometimes} \qquad \mathcal{X} > e^{rT}\mathcal{X}_0.$$

A discount formula illustrates the concept in the simplest form imaginable. If you buy a zero-coupon bond of face 1 for delivery at T in an environment where the rate of interest r is fixed, surely you should pay $\exp(-rT)$. Otherwise one of the parties is a sure winner.

Actually there *are* people earning money in this way. They are known as *arbitrageurs* and operate in terms of very small margins and huge size. It is a fair assumption that their activities wipe out riskless income possibilities, leading to markets *free of arbitrage*. This innocently looking assumption has turned out to have wide reaching consequences and has lead to a tidy theory for derivative pricing. Here is a second, less trivial example.

The price of forward contracts
Forward contracts are arrangements for which the price of future transactions is fixed beforehand. Suppose a financial asset is to be delivered at time $T = Kh$ in exchange for an agreed payment $V_0(K)$[1]. What price V_0 should be charged for such a deal today? It may be mildly surprising that there exists a solution that do not take future market uncertainty into account at all. The argument is based on no arbitrage and runs like this.

Start by buying the asset (which costs V_0) and finance it through the sale of zero-coupon bonds with face one maturing at T. Let $P_{0:K}$ be the price of one such bond. To cover the amount V_0 the number we have sold is $V_0/P_{0:K}$. At $T = Kh$ where all contracts expire we collect the agreed $V_0(K)$ for the asset and have to transfer $V_0/P_{0:k}$ to the bond holders. The net value of these operations is

$$\underset{\textit{asset sale}}{V_0(K)} \quad - \quad \underset{\textit{bond repayments}}{V_0/P_{0:K}}$$

There has been *no* risk at any time. Surely the net value is zero. If so, we have proved that

$$V_0 = V_0(K)P_{0:K}, \tag{1.1}$$

[1]This is a notation we shall use for forward forward contacts with $V_k(K)$ being a price agreed at time t_k for an asset traded at time $T = Kh$.

as anything else would imply profit from risk-less investments (that could be scaled up to enormous proportions!). It is precisely such things that the no arbitrage assumption rules out. Note that the actual price of the underlying asset at expiry is irrelevant[2]. Also note that the argument applies to any liquid asset and that no assumption is made on the future movements of bond prices and interest. The relationship (1.1) will be crucial for pricing interest rate derivatives in Section 14.6.

Binomial movements
A useful exercise is to analyse the consequences of no arbitrage for assets moving **binomially**. By this is meant that the return (at $T = 1$ say) is either

$$R = e^{r_l} - 1 \quad \text{or} \quad R = e^{r_u} - 1 \qquad \text{where} \qquad r_l < r < r_u. \tag{1.2}$$

Excluding other possibilities may appear highly artificial, yet not only is it a useful illustration of the main idea, but it could also be the first step in a more general construction; see below. The pay-off $X = H(R)$ of a derivative of such an asset is either

$$H_l = H(e^{r_l} - 1) \qquad \text{or} \qquad H_u = H(e^{r_u} - 1),$$

and the issue is the value V_0 of such a contract at the time it is set up.

To answer this question consider an *additional* portfolio consisting of investments in the risky asset and a risk-less bank account. If weights are w_1 and w_2, the portfolio is worth $w_1 + w_2$ at the start and grows at the end of the period to

$$w_1 e^{r_l} + w_2 e^r \qquad \text{or} \qquad w_1 e^{r_u} + w_2 e^r.$$

There are just these two possibilities, and they move with the derivative exactly if w_1 and w_2 are determined from

$$w_1 e^{r_l} + w_2 e^r = H_d \qquad \text{and} \qquad w_1 e^{r_u} + w_2 e^r = H_u. \tag{1.3}$$

If you try to solve these equations (we do it below), you will immediately discover that it works because $r_l < r < r_u$. The portfolio is a known as a **replicating** one.

Consider the balance sheet of an institution which is responsible for the orginal derivative and has set up the replicating portfolio on the side. It looks like this:

Value derivative	$-V_0$	$-H_l$	or	$-H_u$
Value replicate	$w_1 + w_2$	H_l	or	H_u
	start		*end of period*	

Such a strategy of being long in a carefully constructed portfolio to compensate for the risk in the derivative leads to net risk zero! Whatever happens there is no *net* payment at the end of the period. But then net payment must be zero even in the beginning, as there would be risk-less income for someone otherwise. It follows that $V_0 = w_1 + w_2$ which can be determined by solving the equations (1.3). Rewrite them as

$$w_1(e^{r_l} - e^r) + (w_1 + w_2)e^r = H_l \qquad \text{and} \qquad w_1(e^{r_u} - e^r) + (w_1 + w_2)e^r = H_u.$$

[2]It could be worth more or less than $V_0(K)$ at that time.

Then eliminate w_1 and deduce that

$$V_0 = e^{-r}\left(\frac{e^{r_u}-e^{r}}{e^{r_u}-e^{r_l}}H_l + \frac{e^{r}-e^{r_l}}{e^{r_u}-e^{r_l}}H_u\right), \tag{1.4}$$

which has an interesting interpretation, discussed next.

Risk neutrality

The value V_0 under binomial movements can be rewritten in a highly suggestive way as

$$V_0 = e^{-r}(q_l H_l + q_u H_u), \tag{1.5}$$

where

$$q_l = \frac{e^{r_u}-e^{r}}{e^{r_u}-e^{r_l}} \qquad \text{and} \qquad q_u = \frac{e^{r}-e^{r_l}}{e^{r_u}-e^{r_l}}. \tag{1.6}$$

Clearly $q_l + q_u = 1$, and condition (1.2) right also ensures that both quantities are positive. Hence, (q_l, q_u) is a probability distribution, usually denoted Q. The second factor in (1.5) is an expectation with respect to this model, and the option premium can therefore be re-expressed as

$$V_0 = \underset{\textit{discounting}}{e^{-r}} \times \underset{\textit{expected pay-off}}{E_Q\{H(R)\}}, \tag{1.7}$$

which calls for a number of comments.

Discounting is obvious (payments under the option is for the future and premium is charged today). The interesting factor is the second one. The probability distribution Q is called the **risk-neutral** model (or measure). Since R is either $\exp(r_l)-1$ or $\exp(r_u)-1$, it follows that

$$E_Q(R) = q_l(e^{r_l}-1) + q_u(e^{r_u}-1) = e^{r}-1$$

after inserting (1.6) for q_l and q_u. In other words, the expected return under the risk-neutral model grows exactly as an ordinary bank account. Why charging the expected pay-off under *this* model? The answer is that being responsible for the option entails no risk, and valuation is therefore in terms of a model within which there is no payment for risk. Note that the probabilities of R jumping up and down in the real market were not mentioned at all.

Of course, the situation is highly artificial. In practice assets change value in an infinite number of ways, not just two. What is remarkable is that the entire construction generalizes, including the pricing formula (1.7), which becomes valid generally if the replicating operation is repeated continously. It is a little like sums of binomial variables becoming normal in the limit. You will find this approach in many textbooks. The line pursued here is different and closer to the actual hedging operations in the market. We shall then end up with a risk-neutral model of the form

$$R = \exp\left(r - \frac{1}{2}\sigma^2 + \sigma\varepsilon\right) - 1 \qquad \text{where} \qquad \varepsilon \sim N(0,1),$$

which gain satisfies $E_Q(R) = \exp(r) - 1$ if you recall the formula for log-normal means (Chapter 2).

1.3 Options on equity

Introduction
Derivatives in equity are in financial literature often written in terms of so-called strikes rather than guranteed returns. Put and call options are then

$$\underbrace{X_P = \max(A - S, 0)}_{\textit{put option}} \qquad \text{and} \qquad \underbrace{X_C = \max(S - A, 0)}_{\textit{call option}}, \tag{1.8}$$

where S is the terminal value of shares priced as S_0 in the beginning. The barrier A is known as the **strike**. With put options there is a right to sell at the price A, and the option holder chooses to do so if $A > S$, gaining $A - S$. The owner is protected against a downside of his stock. Calls are the opposite. Now there is a right to *buy* at an agreed price, an option exercised if the market value is higher.

The link to the earlier version in terms of returns R is simple. Since $S = (1 + R)S_0$, it follows that

$$A - S = A - (1 + R)S_0 = S_0\left(\frac{A - S_0}{S_0} - R\right)$$

and for puts

$$X_P = \max(A - S, 0) = S_0 \max(r_g - R, 0) \qquad \text{where} \qquad r_g = \frac{A - S_0}{S_0}. \tag{1.9}$$

Guaranteed returns and strikes is the same thing with a simple connection between them. The argument for calls is similar.

The pricing theory will be developed in terms of S rather than R. Let S_k be the value of the underlying asset at time $t_k = kh$ for $k = 0, 1, 2 \dots K$ where $S_0 = s_0$ is at present (and known) and the rest belong to the future. The purpose of this section is to review some main types of options and apply simple arbitrage arguments to deduce some immediate properties regarding their value. Pricing through hedging is dealt with in the ensuing sections.

Types of contracts
Derivatives may be classified according to whether or not they expire at some fixed point $T = Kh$. They are called **European** if they do. The pay-off function is then of the form

$$X = H(S_K)$$

where K (or equivalently T) is often referred to as the point of **maturity**. The puts and calls above were of this type as was the cliquet option treated in Section 3.7.

Derivatives which are not European are *path-dependent*; i.e. the pay-off is now influenced by the the entire sequence $S_1, \dots, S_K$. Variations are enormously many. An important type is the so-called **American** analogies to the European calls and puts where the option holder may decide to sell or buy at any time of his choosing. For example, if a call is exercised at t_k, he gains $S_k - A$. The pricing of American style options may seem more difficult than their European counterparts since there is a decision rule when to sell or buy involved in their valuation. It turns out that there is a simple solution for the call, but not for the put; see below. American puts and calls are among the most important equity options of all.

So-called **exotic** versions employ other functions of the path $S_0, S_1, \ldots S_K$. Often mentioned in the literature is the **Asian** type for which

$$X = H(\bar{S}_K) \qquad \text{where} \qquad \bar{S}_K = \frac{1}{K-L}(S_{L+1} + \ldots + S_K).$$

is the average from t_{L+1} up to $T = t_K$. By varying the function H we obtain Asian calls, Asian puts and even Asian cliquets. Numerous other forms have been invented, and quite a lot is known about their pricing.

Valuation: A first look

We shall only be concerned with European options. Their concrete valuation is a subtle theme and the topic of the next section (Section 14.6 too), but what, exactly, can be said at the outset through purely qualitative reasoning? Suppose the contract is drawn up at $t = 0$. A European option must then be worth $V_0 = V(s_0, T)$ where $S_0 = s_0$ is the value of the underlying asset at that time. Note that the earlier history prior to $t = 0$ is absent. That is surely reasonable if asset fluctuations are described through Markov models (as we assume), because all information about future movements then resides in s_0.

Next, what happens during the life of the option? At time t the value of the asset has changed to s say and now the time to expiry is $T - t$. The value must have become $V(s, T-t)$ using the same function $V(s,T)$ as in the beginning. This must be so since, apart from the underlying asset and the time to maturity having changed, the situation remains what it was (invoke the Markov condition again). Later the value will be followed over the time sequence $t_k = kh$ where it will be written

$$V_k = V(S_k, T - t_k) \qquad \text{with} \qquad V_K = H(S_K). \tag{1.10}$$

At expiry the option coincides the pay-out function, but prior to that it is (as yet) a mystery what price we should charge for it if traded and by how much we should allow it to enter our balance sheet. What we *can* assume is that the function $V(s,T)$ is smooth; i.e. that small changes in s or T can't imply more that small changes in V. The precise mathematical meaning that will be attached to this is that the value function is twice differentiable in s and once in T. Why that is needed will emerge in the next section.

Of course, the value of the option also depends on the strike A, the risk-free interest r and parameters in the model for the underlying asset, but there is no point in introducing those fixed quantities into the notation. The central point is that we have to portray value *fluctuations* as the financial market evolves.

The put-call parity

It is possible to gain some insight into valuation through simple arbitage arguments similar to those in the preceding section. One particularly useful deduction is a simple connection between the values of European calls and puts, known as their **parity** relationship. To derive this important result let $V_P(s,T)$ and $V_C(s,T)$ be prices for put and call. Consider a portfolio where we have purchased a (European) call and is responsible for a (European) put. There is also a short position in the underlying stock and an amount of cash. The balance sheet in the beginng and in the end is as follows:

	at $t_0 = 0$	*at* $t_K = T$
European call	$V_C(S_0, T)$	$\max(S_K - A, 0)$
- European put	$-V_P(S_0, T)$	$-\max(A - S_K, 0)$
- underlying asset	$-S_0$	$-S_K$
Cash	$\exp(-rT)A$	A
Portfolio value	?	0

At expiry $T = Kh$ assets possess the values shown. The first and the second line follows directly from the definitions of calls and puts. Whether $S_K < A$ or $\geq A$, it is easy to check that the value of the total portfolio at expiry is exactly zero. But then the value must zero at the start, as well, as otherwise an arbitrage possibility is created. It follows that

$$V_C(S_0, T) + \exp(-rT)A = V_P(S_0, T) + S_0, \tag{1.11}$$

which shows that pricing European puts and calls amounts to the same thing.

A first look on calls

The parity relation produces a useful lower bound on the value of an European call option. Note that $P(s, T) \geq 0$; holding a put option can not be worse than nothing. Hence, from (1.11)

$$V_C(S_0, T) \geq S_0 - \exp(-rT)A. \tag{1.12}$$

Remarkably, this simple result solves the problem of evaluating *American* call options. Recall that the difference when compared with the European is in the freedom of terminating the contract. This means that the value, say $C^{\text{am}}(s, T)$ of an American call is as least as large as the European. Hence, from (1.12)

$$V_C^{\text{am}}(S_0, T) \geq S_0 - \exp(-rT)A > S_0 - A,$$

since $r > 0$. But the right hand side is the amount we receive by exercising the option at $t_0 = 0$, which can't be advisable since the value of our American option is higher.

In other words, liquidating an American call option early, creates an arbitrage opportunity for the opposite party. In a world where arbitrage are absent, American calls will never be exercised early and **should be priced as European calls**. This does not quite hold for didvidend-paying stock, see Hull (2002), chapter 7.

A first look on puts

Clearly $V_C(s, T) \geq 0$; an derivative contact is always worth *something* (not less than zero, at any rate). It follows from the parity relation (1.11) that

$$V_P(S_0, T) \geq S_0 - A\exp(-rT), \tag{1.13}$$

similar to (1.12) for the call.

But unlike above we can not from this lower bound deduce the price of American *puts*. Indeed, the situation is known to be more complicated than for American *calls*. If the prices of the assets are low enough, it *is* profitable to cash in on an American put. Closed pricing formulas, similar to those we shall derive later in this chapter are not available for American puts. These technical complications are beyond the natural limit of this introductory text; a good exposition can be found in Hull (2002).

1.4 Hedging and valuation

Introduction

The binomial model in Section 14.2 produced explicit pricing of derivatives without specifying any probabilities at all (the risk-neutral model was *derived*). This is a forerunner of what is to come, although more realistic situations do require probabilities. With equity options pricing is invariably based on the geometric random walk introduced in Section 5.6. It is then assumed that

$$S_{k+1} = S_k \exp(\xi h + \sigma\sqrt{h}\,\varepsilon_k) \qquad \text{where} \qquad \varepsilon_k \sim N(0,1), \tag{1.14}$$

for $k = 0, 1, \ldots$. As usual the sequence $\{\varepsilon_k\}$ is an independent one. Note that the mean and variance in (1.14) now are ξh and $\sigma^2 h$. Why the model has that form was explained in Section 5.6. An immediate consequence is that the random term (proportional to $\sqrt{h}$) dominates for small h. This has profound influence on option theory. We shall below run hedging operations at each $t_k = kh$ and let $h \to 0$. The random effects are then the most important ones and must be analysed more thoroughly than the others. When $h \to 0$, the model is often called **geometrical Brownian motion**.

The purpose of this section is to examine hedging and its consequences in an intuitive way and demonstrate why such opportunites influence prices in a liquid market (where there buyers and sellers for everything). A hedging process will be simulated in the computer at the end enabling us to see what happens in concrete terms. Much of the detailed mathematics is deferred to Section 14.5

Actuarial and risk-neutral pricing

Consider the following two valuations of a future pay-off $X = H(S_K)$ at time $t_0 = 0$ where $S_0 = s_0$:

$$\underset{\textit{actuarial}}{V_0 = e^{-rT}E\{H(S_K)|s_0\}} \qquad \textit{and} \qquad \underset{\textit{Q-model, risk-neutral}}{V_0 = e^{-rT}E_Q\{H(S_K)|s_0\}} \tag{1.15}$$

Uncertainty at expiry is influenced by the value of the underlying asset to-day (hence the *conditional* means), and there is a discount. The ordinary actuarial valuation on the left is under the model assumed. In insurance that lead to a break-even situation with neither party gaining or losing. But now this is no longer so since the real risk, reduced though hedging, is lower. Indeed, when the entire process is simulated in the computer in Figure 14.1 below, we shall see that actuarial pricing would entail very high profits for the seller.

Taking hedging into account will eventually lead to evaluation under the risk-neutral Q-model on the right in (1.15). The old ξ is then replaced by

$$\xi_q = r - \sigma^2/2 \qquad \text{(risk-neutral } \xi\text{)}, \tag{1.16}$$

but the model is otherwise the same as it was, with the same volatility for example. One consequence is that *the price does not depend on the real ξ at all.* That is fortunate since ξ is very hard to pin down (Section 2.2). The actual price is then the question of recalling from the discussion in Section 5.6 that

$$S_K = s_0 \exp\{T(r - \sigma^2/2) + \sqrt{T}\sigma\varepsilon\} \qquad \text{(under Q-model)},$$

where $\varepsilon \sim N(0,1)$, and we may calculate the right hand side of (1.15). In the general case this may require simulation (or numerical integration), but puts and calls admit closed expressions. Those are

$$V_P(s_0,T) = e^{-rT}A\Phi(a) - s_0\Phi(a-\sigma\sqrt{T}), \qquad \text{(European put)} \tag{1.17}$$

$$V_C(s_0,T) = s_0\Phi(-a+\sigma\sqrt{T}) - e^{-rT}\Phi(-a) \qquad \text{(European call)} \tag{1.18}$$

where

$$a = \frac{\log(A/s_0) - rT + \sigma^2 T/2}{\sigma\sqrt{T}} \tag{1.19}$$

The formula for the put is adapted from (**??**) in Section 3.7, and the call from the parity relation (1.11); see Exercises ?? and ?? for details. The volatility σ is typically taken from observed trading through the so-called **implied view**. Prices observed is then entered on the left in (1.17) or (1.18) and solved for σ.

The hedge portfolio and its properties

Hedging must start with an understanding of how the values $\{V_k\}$ of an option evolves. Suppose $V_k = V(S_k, T-t_k)$ where $V(s,T-t)$ (which will be determined eventually) is a smooth function of s and t that can be approximated by an ordinary Taylor series around S_k and t_k. With the required terms included this yields

$$V(s,T-t) \doteq V(S_k,t_k) + \Delta_k(s-S_k) + \frac{1}{2}\Gamma_k(s-S_k)^2 + \Theta_k(t-t_k)$$

where

$$\Delta_k = \frac{\partial V(S_k,T-t_k)}{\partial s}, \qquad \Gamma_k = \frac{\partial^2 V(S_k,T-t_k)}{\partial s^2} \quad \text{and} \quad \Theta_k = \frac{\partial V(S_k,T-t_k)}{\partial t}. \tag{1.20}$$

Note the unequal treatment of s and t. Inserting $s = S_{k+1}$ and $t = t_{k+1}$ yields the approximation

$$V_{k+1} - V_k \doteq \underbrace{\Delta_k(S_{k+1}-S_k)}_{\textit{order } \sqrt{h}} + \underbrace{\tfrac{1}{2}\Gamma_k(S_{k+1}-S_k)^2}_{\textit{order } h} + \underbrace{\Theta_k(t_{k+1}-t_k)}_{\textit{order } h}, \tag{1.21}$$

where the magnitude of the three contributions on the right has been indicated (more on the details in the next section). To capture contributions of order h two terms are needed in s and only one in t. A full Taylor series require an infinite number. Terms neglected are all of order $h^{3/2}$ and smaller.

Consider now the hedge portfolio indicated in Section 14.1 where the option seller at time t_k buys $\Delta_k S_k$ of the underlying asset. When the the obligation for the option is added, the value of this portfolio is

$$\underbrace{\mathcal{H}_k = -V_k + \Delta_k S_k}_{\textit{value at time } t_k} \qquad \text{and} \qquad \underbrace{\mathcal{H}_{1k} = -V_{k+1} + \Delta_k S_{k+1}}_{\textit{value at time } t_{k+1}}. \tag{1.22}$$

The change from t_k to t_{k+1} is caused by both the derivative and the asset. Now

$$\mathcal{H}_{1k} - \mathcal{H}_k = -(V_{k+1} - V_k) + \Delta_k S_{k+1} - \Delta_k S_k,$$

or after inserting (1.21)

$$\mathcal{H}_{1k} - \mathcal{H}_k \doteq -\frac{1}{2}\Gamma_k(S_{k+1} - S_k)^2 - \Theta_k(t_{k+1} - t_k). \tag{1.23}$$

The dominant first term on the right in (1.21) has disappeared! This is precisely what we sought. Volatilatility in the hedge portfolio is *smaller* than in the option itself.

Financial status over time

To understand the aggregated effect of hedging we have to analyse the financial state of the option seller from beginning to end. In addition to the sequence of values $\{V_k\}$ and $\{\Delta_k S_k\}$ of derivatives and hedges there are also holdings $\{B_k\}$ in cash which will fluctuate as the underlying asset is bought and sold. Suppose the seller collects the option premium upfront and uses parts of it to purchase underlying stock, borrowing if necessary. The cash holding then develops according to the scheme

$$\underset{\textit{at the start}}{B_0 = V_0 - \Delta_0 S_0,} \qquad \underset{\textit{for } k=1,2,\dots K-1}{B_k = e^{rh}B_{k-1} + (\Delta_{k-1} - \Delta_k)S_k,} \qquad \underset{\textit{at expiry}}{B_K = e^{rh}B_{K-1}.} \tag{1.24}$$

At time t_k the equity holding is rebalanced. The seller now possesses $\Delta_{k-1}S_k$ from the previous round and wants $\Delta_k S_k$, leading to a change of the account. There is (of course) no such rebalancing at expiry. The sequence of values $\{\mathcal{X}_k\}$ of the entire portfolio of derivative, asset and cash are then

$$\underset{\textit{at the start}}{\mathcal{X}_0 = 0,} \qquad \underset{\textit{for } k=1,2,\dots,K-1}{\mathcal{X}_k = -V_k + \Delta_k S_k + B_k,} \qquad \underset{\textit{at expiry}}{\mathcal{X}_K = -H(S_K) + \Delta_{K-1}S_K + B_K,} \tag{1.25}$$

written down *after* the underlying asset held has been adjusted for the next period. Of course, there is no rebalancing at expiry where the value of the option and the pay-off $H(S_K)$ is the same.

These relationships will be studied theoretically in the next section. In the present sequel what happens will be examined through simulations. Consider a call option. Its value $V_k = V_C(S_k, T - t_k)$ at time t_k can be adapted from (1.18). This leads to

$$V_{Ck} = S_k\Phi(-a_k + \sigma\sqrt{T - t_k}) - e^{-r(T-t_k)}\Phi(-a_k)$$

where

$$a_k = \frac{\log(A/S_k) - r(T - t_k) + \sigma^2(T - t_k)/2}{\sigma\sqrt{T - t_k}}$$

The hedge Δ_k must be found by differentiating V_{Ck} with respect to S_k. That is straightforward (see Execise 14.? for details) yielding

$$\Delta_k = \Phi(-a_k + \sigma\sqrt{T - t_k}).$$

A simulation is then a question of generating asset development S_k^* through (1.14), calculating option values $V_k^* = C_k^*$ and hedges Δ_k^* by means of the formulas given and finally plugging those into (1.24) and (1.25) to find simulated cash B_k^* and portfolio values $\mathcal{X}_k^*$.

Numerical experiment

The call option portfolio $\mathcal{X}_k$ has been simulated ten times in Figure 14.1. Rebalancing was

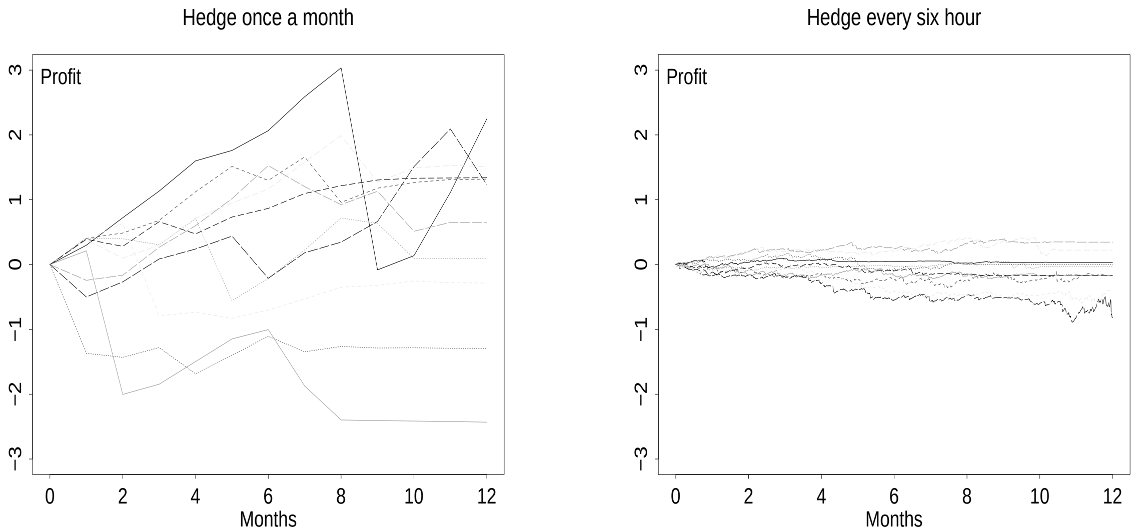


Figure 14.1 Simulated scenarios of a portfolio set up to hedge call options. Rebalancing is daily (left) and monthly (right).

monthly to the left and every six hours to the right[3]. The detailed conditions are as follows:

Equity drift:	10% annually	Time to maturity:	One year
Equity volatility:	25% annually	Option strike:	100
Riskfree interest:	4% annually	Initial value equity:	100

Note that the simulations are run from the true asset model. What we see is that frequent rebalancing leads to a a radical reduction of the risk. There was no money involved at the start and almost none at the end. The spread will vanish completely when we let $h \to 0$ in the next section.

What would have happened if the market had allowed *actuarial* pricing of the derivative? An option seller who knew the ropes would have been able to run the same risk-reducing operations, but would now have collected more money upfront. From (1.36) left we may deduce that the actuarial option premium is

$$S_0 e^{T(\xi+\sigma^2/2-r)}\Phi(-a+\sigma\sqrt{T}) - e^{-rT}\Phi(-a) \qquad \text{where} \qquad a = \frac{\log(A/S_0)-\xi T}{\sigma\sqrt{T}};$$

see Exercise 14.?. This reduces to (1.18) if the risk-neutral $\xi_q = r - \sigma^2/2$ is inserted for ξ. Under the conditions given above option prices become[4]

$$\underset{\textit{actuarial}}{18.334 \ \text{ under } \ \xi = 10\%} \qquad\qquad \underset{\textit{risk-neutral}}{11.837 \ \text{ under } \ \xi = 0.875\%},$$

where the risk-neutral ξ is much smaller than the real one. The actuarial option price would have been about 6.5 higher, and all simulations in Figure 14.1 would have been lifted this amount if the market had been willing to pay that much. That would have lead to a very handsome profit indeed for the option seller with little risk attached.

[3] With 250 days in a financial year this corresponds to rebalancing being carried out $K = 1000$ times.
[4] The risk-neutral ξ is under the parameters assumed $\xi_q = 0.04 - \frac{1}{2} \times 0.25^2 = 0.00875$.

The conclusion is clear and robust. Free markets do not permit that kind of (almost) risk-less profit, and would force pricing based on the *real* risk. Rebalancing costs have been ignored, but they do not move things substantially.

1.5 Mathematics of equity options

Introduction

Hedging and risk-neutral pricing evidently works. No money was put into the scheme in Figure 14.1, and the account $\mathcal{X}_k$ fluctuated appropriately around the origin! But why the particular form of the Q-model, and above all, why is the drift ξ immaterial for pricing? It is the aim of this section to provide answers to both questions. The second one is intuitive. Hedging removes the effect of everything foreseeable, therefore ξ. If that is accepted, the derivation of option prices becomes a simple matter. A bit of mathematics at the end of the section will establish that ξ is indeed irrelevant.

The essential conditions are liquid markets free of arbitrage and dynamic rebalancing of the hedge portfolio at no cost. In the preceding section these operations took place at all time points $t_k = kh$, $k = 0, 1, \ldots, K-1$ right up to the point of maturity. As with elementary Poisson processes in Chapter 8 we may simultaneously let $h \to 0$ and $K \to \infty$ while keeping their product $T = Kh$ fixed. Simplified mathematical expressions then appear in the limit. There will be errors coming from higher order terms in the series expansions, but those can be overlooked if they are of order of magnitude $h^{3/2}$ or smaller. The reason is that there are K contributions to the total error, one for each time the portfolio is rebalanced. Their *combined* effect is of order

$$Kh^{3/2} = T\sqrt{h} \to 0 \quad \text{as } h \to 0,$$

suggesting that the goal should be to identify the terms up to size h, and forget the rest.

The situation at expiry

When the option matures at time $t_K = T$, the option seller has been involved in three different types of transactions. The premium V_0 (by now grown to $\exp(rT)V_0$) was collected originally, then there is a claim $H(S_K)$ to be paid and finally the cumulative effect of all the hedging positions $\Delta_k S_k$ will have produced earning (or loss). A convenient way to express the latter is through

$$G_k = e^{-rh}\Delta_k S_{k+1} - \Delta_k S_k, \tag{1.26}$$

which is the gain from the k'th hedge when discounted back to t_k. When all these transactions are added with appropriate interest, we must obtain the financial state of the option seller at expiry, or

$$\mathcal{X}_K = \underbrace{V_0 e^{rT}}_{\textit{premium received}} - \underbrace{H(S_K)}_{\textit{pay-off}} + \underbrace{\textstyle\sum_{k=0}^{K-1} G_k e^{r(T-t_k)}}_{\textit{accumulated through hedging}}, \tag{1.27}$$

which is obvious from an economic point of view. That's simply how a sequence of transactions adds up.

In case you remain unconvinced, here is a mathematical proof using (1.24) and (1.25). Consider first the cash account B_K, which is (as always) the sum of all deposits/withdrawals with interest. From (1.24) we have

$$B_K = (V_0 - \Delta_0 S_0)e^{rT} + \sum_{k=1}^{K-1}(\Delta_{k-1} - \Delta_k)S_k e^{r(T-t_k)},$$

which in combination with (1.25) right yields

$$\begin{aligned}\mathcal{X}_K &= -H(S_K) + \Delta_{K-1}S_K + B_K \\ &= -H(S_K) + \Delta_{K-1}S_K + (V_0 - \Delta_0 S_0)e^{rT} + \sum_{k=1}^{K-1}(\Delta_{k-1} - \Delta_k)S_k e^{r(T-t_k)}.\end{aligned}$$

Each Δ_k are involved in precisely two terms. When those are identified, the expression for $\mathcal{X}_K$ can be rewritten

$$\begin{aligned}\mathcal{X}_K &= V_0 e^{rT} - H(S_K) + \sum_{k=1}^{K-1}\Delta_k(S_{k+1}e^{r(T-t_{k+1})} - S_k e^{r(T-t_k)}) \\ &= V_0 e^{rT} - H(S_K) + \sum_{k=0}^{K-1}\Delta_k(e^{-rh}S_{k+1} - S_k)e^{r(T-t_k)},\end{aligned}$$

which is (1.27).

Valuation

The option seller has made no investment so a break-even, pure premium would make at expiry

$$E(\mathcal{X}_K|s_0) = 0.$$

In insurance a price for risk (the risk *loading*) would come on top, but here things are arranged so that there is no variability in $\mathcal{X}_K$ at all (at least in theory), making the break-even premium logical on economic grounds. The very same argument will also show that the value of ξ is irrelevant. Indeed, it was absent in all previous formulas; why will be seen below.

Start by applying expectation to all terms in the decomposition formula (1.27). This yields

$$E(\mathcal{X}_K|s_0) = V_0 e^{rT} - E\{H(S_K)|s_0\} + \sum_{k=0}^{K-1}E(G_k|s_0)e^{r(T-t_k)},$$

where V_0 is a constant, fixed by s_0. Equating this with zero leads to

$$V_0 = e^{-rT}E\{H(S_K)|s_0\} - \sum_{k=0}^{K-1}E(G_k|s_0)e^{-rt_k}, \tag{1.28}$$

an interesting identity revealing that the "fair" option price is its expected, *discounted* pay-off at expiry *minus* the expected *discounted* result of all hedge operations. Note that the actual risk of the seller now is taken into account.

The expected gain from the hedging operations may be calculated. By (1.26) and (1.14)

$$G_k = \left(e^{-rh}\frac{S_{k+1}}{S_k} - 1\right)\Delta_k S_k = (e^{-rh+\xi h+\sigma\sqrt{h}\varepsilon_k} - 1)\Delta_k S_k$$

and

$$E(G_k|s_0) = E(e^{-rh+\xi h+\sigma\sqrt{h}\varepsilon_k} - 1|s_0)E(\Delta_k S_k|s_0)$$

since the two factors are stochastically independent. But the standard formula for the mean of log-normal variables now yields

$$E(G_k|s_0) = (e^{h(\xi+\sigma^2/2-r)} - 1)E(\Delta_k S_k|s_0). \tag{1.29}$$

Suppose it is assumed that V_0 does not depend on ξ. That must apply to the right hand side of (1.28) too, and any value of ξ may be inserted. The risk-neutral $\xi = r - \sigma^2/2$ is the most convenient one. Now $E(G_k|s_0) = 0$ by (1.29), and we are left with

$$V_0 = e^{-rT}E_Q\{H(S_k|s_0)\}.$$

The expectation is calculated under the risk-neutral Q, precisely as claimed in (1.15) right.

Why the risk is eliminated

Remaining issues are why hedging makes the variability of the terminal $\mathcal{X}_K$ disappear, and why valuation isn't influenced by the drift ξ. The latter is sometimes regarded as "extremely surprising", but it isn't really. It is certainely plausible that hedging and re-hedging at will removes the impact of anything predictable! To argue the case we must let $h \to 0$ and $K = T/h \to \infty$ and examine the fluctuations under small time incements. There is no extra difficulty in extending the asset model to

$$S_{k+1} = S_k e^{\xi_k h+\sigma_k\sqrt{h}\,\varepsilon_k} \qquad \text{where} \qquad \xi_k = \xi_k(S_k) \text{ and } \sigma_k = \sigma_k(S_k), \tag{1.30}$$

allowing time-varying drift and volatility influenced by the current price of the asset.

The argument is based on the *changes* in the hedge portfolio of the option seller. Introduce

$$D_k = \mathcal{H}_{1k} - e^{rh}\mathcal{H}_k \tag{1.31}$$

where $\mathcal{H}_k$ and $\mathcal{H}_{1k}$ are the values of at t_k and t_{k+1}; see (1.22). Moreover, we also have from (1.25) that

$$\mathcal{X}_k = \mathcal{H}_k + B_k \qquad \text{and} \qquad \mathcal{X}_{k+1} = \mathcal{H}_{1k} + e^{rh}B_k,$$

expressing how the value of $\mathcal{X}_k$ grows with the hedge portfolio and the cash holding. But now

$$D_k = (\mathcal{X}_{k+1} - e^{rh}B_k) - e^{rh}(\mathcal{X}_k - B_k) = \mathcal{X}_{k+1} - e^{rh}\mathcal{X}_k$$

or

$$\mathcal{X}_{k+1} = e^{rh}\mathcal{X}_k + D_k, \quad k = 0, \ldots, K-1 \qquad \text{starting at} \qquad \mathcal{X}_0 = 0.$$

This is from a mathematical point of view the same as an interest-earning bank account with D_k as deposits. At the end X_K is the sum of all contributions with appropriate interest; i.e.

$$\mathcal{X}_K = \sum_{k=0}^{K-1} e^{r(T-t_k)}D_k \qquad \text{where} \qquad D_k = \mathcal{H}_{1k} - e^{rh}\mathcal{H}_k,$$

and D_k must be studied. The central quantity is

$$\delta_k = D_k - E(D_k|s_0) \tag{1.32}$$

with which $\mathcal{X}_K$ can be rewritten

$$\mathcal{X}_K = \sum_{k=0}^{K-1} e^{r(T-t_k)}E(D_k|s_0) \;+\; \mathcal{E}_K \qquad \text{where} \qquad \mathcal{E}_K = \sum_{k=0}^{K-1} e^{r(T-t_k)}\delta_k. \tag{1.33}$$

Here $\mathcal{E}_K$ is a *remainder* term with zero mean (an immediate consequence of all δ_k having zero mean), and it will be shown below that *its variance become zero* as $h \to 0$ (and $K = T/h \to \infty$).

This tells us that as the hedging operations are carried out more and more frequently all uncertainty is disposed of. In the limit the *the entire sequence* $\delta_0, \dots, \delta_{K-1}$ becomes zero, and all D_k are to equal their expectations. But surely those must then be zero! If they are not, the value of the *risk-less* hedge portfolio $\mathcal{H}_k$ would grow differently from the *risk-less* rate and that would be arbitrage. It follows that all $D_k = 0$ and $\mathcal{X}_K = \mathcal{E}_K$ in (1.33). But both mean and variance of $\mathcal{X}_K$ then vanish in the limit (as indicated in Figure 14.1), supporting the pricing scheme used above.

Why the drift is eliminated too

To understand why the drift ξ is absent in the valuation formulas we have to examine the hedging operations in more detail. Recall from (1.23) that

$$\mathcal{H}_{1k} - \mathcal{H}_k \doteq -\frac{1}{2}\Gamma_k(S_{k+1} - S_k)^2 - \Theta_k(t_{k+1} - t_k),$$

and (1.30) is needed to examine the the first term on the right. Take $x = \xi_k h + \sigma_k\sqrt{h}\,\varepsilon_k$ in the approximation $e^x - 1 \doteq x$ (accurate for small x), and deduce that

$$S_{k+1} - S_k = (e^{\xi_k h + \sigma_k\sqrt{h}\,\varepsilon_k} - 1)S_k \doteq (\xi_k h + \sigma_k\sqrt{h}\,\varepsilon_k)S_k.$$

This yields

$$(S_{k+1} - S_k)^2 \doteq h\sigma_k^2\varepsilon_k^2 S_k^2,$$

having disposed of terms of order $h^{3/2}$ and smaller. It follows that

$$\mathcal{H}_{1k} - \mathcal{H}_k \doteq -h(\frac{1}{2}\Gamma_k\sigma_k^2 S_k^2\varepsilon_k^2 + \Theta_k)$$

so that

$$D_k = \mathcal{H}_{1k} - e^{rh}\mathcal{H}_k \doteq (\mathcal{H}_{1k} - \mathcal{H}_k) - rh\mathcal{H}_k \;\;\doteq\;\; \underbrace{-h(\tfrac{1}{2}\Gamma_k\sigma_k^2 S_k^2 + \Theta_k + r\mathcal{H}_k)}_{\textit{fixed given } S_k} \;+\; \underbrace{\delta_k}_{\textit{random}}$$

where

$$\delta_k = \frac{h}{2}\Gamma_k S_k^2\sigma_k^2(\varepsilon_k^2 - 1). \tag{1.34}$$

Note that δ_k is the same quantity as in (1.32), though we haven't proved that yet. It will follow when we verify that $E(\delta_k) = 0$.

The crucial point is that whereas the volatility σ_k is present in the approximations for D_k, the drift ξ_k has dropped out. This tells us that that *we do not have to know ξ_k to carry out the hedge*. True, the fluctuations $\{S_k\}$ of the asset do depend on $\{\xi_k\}$, but as long as we progress as above our operations are risk-free regardless. But then the option premium collected in the beginning can't depend on ξ_k. If it did, we would collect the fee for a set $\{\xi_k\}$ the market believes in and set up the hedge process for one of the cheaper ones. There is then a surplus over $\mathcal{X}_0 = 0$ in the beginning which carries through until the end *with no risk*, violating the basic assumption of no arbitrage.

An auxiliary result

What remains is the study of δ_k in (1.34). It is convenient notationally to rewrite as

$$\delta_k = hY_k\eta_k \qquad \text{where} \qquad Y_k = \frac{1}{2}\sigma_k^2 S_k^2 \Gamma_k \quad \text{and} \quad \eta_k = \varepsilon_k^2 - 1.$$

Since $\varepsilon_0, \dots, \varepsilon_{K-1}$ are independent and Normal $(0,1)$, it follows that $\eta_0, \dots, \eta_{K-1}$ are also independent with

$$E(\eta_k) = 0 \qquad \text{and} \qquad E(\eta_k^2) = 2;$$

see Appendix A for the latter. Moreover, all preceding $Y_1, \dots, Y_k$ must be independent of η_k. It is under these circumstances simple to study the accumulated effect of δ_k. Start by noting that

$$E(\delta_k) = E\{hY_k\eta_k\} = hE(Y_k)E(\eta_k) = 0$$

and on a similar argument

$$\text{var}(\delta_k) = E(\delta_k^2) = E\{h^2 Y_k^2 \eta_k^2\} = h^2 E(Y_k^2)E(\eta_k^2) = 2h^2 E(Y_k^2).$$

Moreover, the entire sequence $\delta_0, \dots, \delta_{K-1}$ is uncorrelated. To see this, suppose $i > k$. Then

$$\text{cov}\{\delta_k, \delta_i\} = E(\delta_k\delta_i) = E\{hY_k\eta_k hY_i\eta_i\} = h^2 E\{Y_k\eta_k Y_i\}E(\eta_i) = 0.$$

Let's go back to the remainder $\mathcal{E}_K$ in (1.33) right. The ordinary sums of expectation and variance formulas yield

$$E(\mathcal{E}_K) = 0 \qquad \text{and} \qquad \text{var}(\mathcal{E}_K) = \sum_{k=0}^{K-1} \text{var}\left(e^{r(T-t_k)}\delta_k\right) = 2h^2 \sum_{k=0}^{K-1} e^{2r(T-t_k)}E(Y_k^2),$$

and the remaining question is why the variance tends to zero as $h \to 0$ and $K = T/h$.

Here is an informal argument. Both σ_k and Γ_k are bounded above by some constant, the former because anything else is nonsensical and the latter becuse it is the second partial derivative of the option price, therefore a smooth function of (s,t), and hence bounded over a bounded region. Since the fourth order moment of the log-normal S_k is bounded too, there must exist a constant D (not depending on k) so that

$$e^{2r(T-t_k)}E(Y_k^2) = e^{2r(T-t_k)}\frac{1}{2}E(\sigma_k^2\Gamma_k^2 S_k^4) \leq D.$$

But then

$$\text{var}\left(\sum_{k=0}^{K-1} e^{r(T-t_k)}\delta_k\right) = 2h^2 \sum_{k=0}^{K-1} e^{2r(T-t_k)}E(Y_k^2) \leq 2h^2 KD = 2hTD,$$

and the right hand side tends to zero as $h \to 0$. This tells us that $\text{var}(\mathcal{E}_K) \to 0$.

1.6 Interest rate derivatives

Introduction

Derivatives in the money market are no less important than those for equity. Their duration is longer, up to decades (whereas equity options typically expire within a year). As an example, consider an interest rate **floor**. This is a series of put agreements (known as **floorlets**) under which the option holder is reimbursed when the floating rate of interest falls below some barrier r_A. At $t_k = kh$ there will be a pay-off

$$X_k = \max(r_A - r_k, 0)v_0, \tag{1.35}$$

where v_0 is the the amount of capital involved. Note that r_A and r_K now apply to an interval of fixed length h. Interest rates are in this (and next) section *not* compounded continuously.

A floor is an interest rate guarantee, providing compensation whenever $r_K < r_A$. Such an instrument could be used by indviduals and institutions seeking a certain minimum return on their investments in the money market. Whether that is a sensible strategy or not depends. The price for equity options in Section 3.7 was quite steep, but conditions are now altered. There is a huge number of money market risk-reducing instruments. The **cap** is the call-like opposite of the floor. Now a borrower worries about high future floating rates of interest and seeks compensation when that happens. This and other common types will be reviewed below. The obvious question is valuation. How much should money market derivatives cost? Hedging is again a key issue, but the argument is not the same as with equity. The crucial difference is that interest rate is not a commodity that can be purchased as a hedge. Instead the seller of such derivatives must reduce risk by investing in *other* money market products. The theory is more varied and less settled than for equity, but there *is* a standard approach leading to the pricing used in practice. That is the line presented.

Martingale pricing

The martingale perspective unifies equity and money market derivatives. Options in equity have been priced through

$$V_0 = P_{0:K} E_Q\{H(S_K)|s_0\}$$

where the discount (instead of the earlier e^{-rT}) now is written as $P_{0:K}$. Here $P_{k:K}$ is the price at t_k of a bond of face one expiring at $T_K = T$. In particular, $P_{K:K} = 1$. Introduce

$$M_k = \frac{V_k}{P_{k:K}} \qquad \text{and} \qquad M_K = \frac{V_K}{P_{K:K}} = H(S_K),$$

and it follows that the price equation may be re-expressed

$$M_0 = E_Q(M_K|s_0) \qquad \text{or for general } k \qquad M_k = E_Q(M_K|s_k).$$

Stochastic processes $\{M_k\}$ satisfying the condition on the right is known as **martingales**. Their properties are much studied in the probabilistic literature; see Section 14.8, but not much of that will be of direct use here.

This martingale edifice was derived through hedging and no arbitrage, and *it applies to money market derivatives* too, although the detailed theory is different. An (uncertain) value V_K of some asset in the money market now takes the place of the pay-off $H(S_K)$, and the pricing equation becomes

$$V_0 = P_{0:K} E_Q(V_K), \tag{1.36}$$

where Q is again a risk-neutral model. Note that the information available at time $t_0 = 0$ is *not* made visible in the mathematical notation (as was done with equity). All valuation of money market derivatives below is carried out through (1.36). Justification through hedging and no arbitrage is given in the next section.

Forward contracts and prices

The pricing of interest derivatives makes use of forward contracts observed in the market. Suppose V_K is traded at $T = Kh$ for a price $V_0(K)$ agreed at $t_0 = 0$. That situation was analysed in Section 14.2 and lead to

$$V_0 = P_{0:K} V_0(K)$$

as the value of the contract at $t_0 = 0$; see (1.1). But when comparing with (1.36) it emerges that

$$E_Q(V_K) = V_0(K), \tag{1.37}$$

and the expected value of V_K under risk neutrality can be taken from forward prices $V_0(K)$ quoted in the market.

As an example, consider the floorlet in (1.35). We need the distribution of the floating rate of interest r_k under risk neutrality. The forward rate $r_0(k)$ applies to same the period (from t_{k-1} to t_k), but is settled today, and its information is available. What (1.37) relates is that

$$E_Q(r_k) = r_0(k). \tag{1.38}$$

This will go into floor, cap and swap pricing below.

There are many other examples of forward money market products. Indeed, the modern financial world has erected a veritable system of them. A bond maturing at $T_{K_2} = K_2 h$ may be sold at $T_{K_1} = K_1 h$ for a price $P_0(K_1 : K_2)$ agreed at time $t_0 = 0$. There are forward rates of interest $r_0(K_1 : K_2)$ with a similar meaning (intimately connected to forward bonds, of course). Note the notation where the parenthesis define the period and the subscript the time of the agreement[5]. An almost bewildering number of interest quantities have now been introduced. There are the floating rate of interest (r_k), prices of unit face zero-coupon bonds ($P_{0:k}$), the money market yield ($y_{0:k}$, see Section 6.4) and forward quantities such as $r_0(K)$ and $P_0(K)$. Between them there are a number of mathematical relationships, all a consequence of no arbitrage.

Relevant for the derivative pricing is the connection between bond prices and forward rates. Consider a bond expiring at $t_k = kh$. At $t_0 = 0$ its value is $V_0 = P_{0:k}$, and its forward price $V_0(k-1) = 1/\{1 + r_0(k)\}$. Hence from (1.1)

$$\underbrace{P_{0:k}}_{\textit{value at } t_0=0} = P_{0:k-1} \cdot \underbrace{\frac{1}{1+r_0(k-1)}}_{\textit{forward value}},$$

[5] Also note that $P_0(K) = P_0(K-1:K)$ and $r_0(K) = r_0(K-1:K)$.

which yields

$$r_0(k) = \frac{P_{0:k-1}}{P_{0:k}} - 1 \qquad \text{and also} \qquad r_0(k) = \frac{(1+y_{0:k})^k}{(1+y_{0:k-1})^{k-1}} - 1, \tag{1.39}$$

where the second relationship is due to $P_{0:k} = 1/(1+y_{0:k})^k$; see Section 6.4. Forward rates of interest can be calculated from current bond prices or (equivalently) the yield curve. A numerical example is given in Figure 14.2 below. A similar result for forward bonds is

$$P_0(K_1:K_2) = \frac{P_{0:K_2}}{P_{0:K_1}} \qquad \text{and} \qquad P_0(K_1:K_2) = \frac{(1+y_{0:K_1})^{K_1}}{(1+y_{0:K_2})^{K_2}} - 1, \tag{1.40}$$

which are both based on the same argument.

Floors and caps

Floors and caps are among the most important derivatives in the money market. Their pay-offs at time t_k are

$$\underbrace{X_{Fk} = \max(r_A - r_k, 0)v_0}_{\textit{floorlet payment}} \qquad \underbrace{X_{Ck} = \max(r_k - r_A, 0)v_0,}_{\textit{caplet payment}} \tag{1.41}$$

which usually apply for many periods k under a single contract. Introduce

$$\underbrace{V_{Fk} = P_{0:k}E_Q(X_{Fk})}_{\textit{floorlet price}} \qquad \text{and} \qquad \underbrace{V_{Ck} = P_{0:k}E_Q(X_{Ck})}_{\textit{caplet price}} \tag{1.42}$$

as the upfront floorlet and caplet prices, between which there is the parity relationship

$$V_{Fk} - V_{Ck} = P_{0:k}\{r_A - r_0(k)\}v_0. \tag{1.43}$$

Only the floorlet price V_{Fk} will be specified below.

To prove (1.43), note that

$$X_{Fk} - X_{Ck} = (r_A - r_k)v_0,$$

which implies that

$$E_Q(X_{Fk}) - E_Q(X_{Ck}) = E_Q(X_{Fk} - X_{Ck}) = E_Q(r_A - r_k)v_0 = \{r_A - E_Q(r_k)\}v_0.$$

Since $r_0(k) = E_Q(r_k)$, it follows that

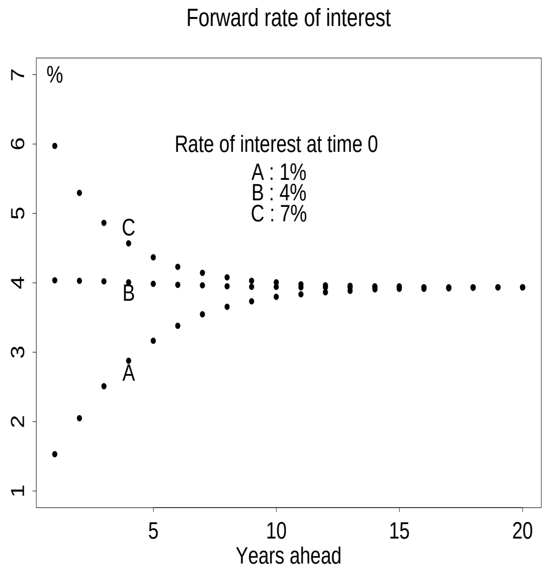
$$E_Q(X_{Fk}) - E_Q(X_{Ck}) = \{r_A - r_0(k)\}v_0$$

and multiplying with $P_{0:k}$ yields (1.43).

We need a risk-neutral model for the floating rate r_k. The standard approach is to use the log-normal which to make $r_0(k) = E_Q(r_k)$ must be of the form

$$r_k = r_0(k)e^{-\sigma_k^2/2+\sigma_k\varepsilon_k} \qquad \text{where} \qquad \varepsilon_k \sim N(0,1).$$

Note that a specific dynamic model is *not* needed. All the linear ones considered in the preceding chapter will do when applied on logarithmic scale. The underlying dynamics will influence how the volatility σ_k vary with k, but they might in practice also be found from



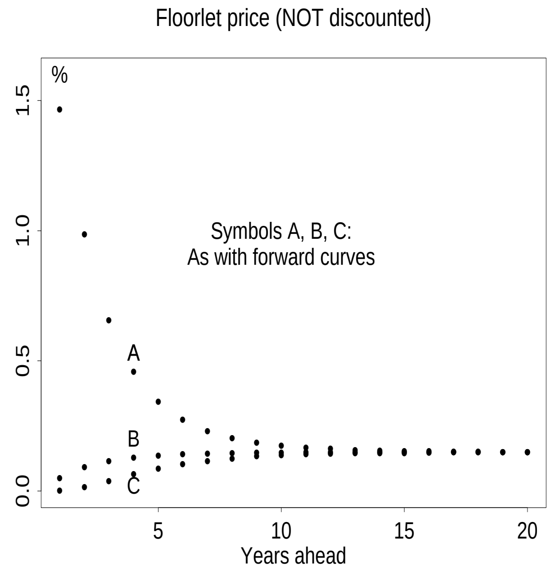


Figure 14.2 Forward rates of interest (left) and corresponding floorlet prices (right) under the conditions described in the text.

market information ("the implied view"). With σ_k known from one source or another, the detailed calculation is the same as for equity. We have to evaluate

$$E\{\max(r_A - r_k, 0)\}v_0.$$

under the model assumed. This results in the floorlet price

$$V_{Fk} = P_{0:k}\{r_A\Phi(-a_k + \sigma_k) - r_0(k)\Phi(-a_k)\}v_0 \tag{1.44}$$

where

$$a_k = \frac{\log(r_0(k)/r_A) + \sigma_k^2/2}{\sigma_k}; \tag{1.45}$$

see Exercise 14.? for mathematical details. The upfront price for the entire floor is found by adding over the individual floorlets. For caps and caplets see Exercise 14.?.

Numerical example

How expensive is a floor agreement? One of the quantities entering is the forward rate of interest $r_0(k)$. In practice that is taken from market information, but here theoretical values have been computed under the Black-Karisinsky model of Section 5.7. The forward rate was converted from the yield curve; see (1.39), which was in turn obtained by the the procedure in Section 6.4; see Algorithm 6.1 in particular. Although the details there were given for the Vasiček model, they are not hard to modify. The parameters of the Black-Karisinsky model were taken as

$$\xi = 4\%, \qquad \sigma = 0.25, \qquad a = 0.7,$$

and three values of the *initial* rate of interest were tried; i.e. 1% ("low", case A), 4% ("medium", case B) or 7% ("high", case C). The resulting forward rates are plotted in Figure 14.2 left. Note the impact of the initial state of the economy. In the beginning the forward rates is strongly influenced by the start, but after a while (roughly a decade) the reversion to mean effect have forced all of them to coincide with the long term average $\xi = 4\%$.

Guarantee and risk-neutral volatilities for the floorlet were

$$r_A = 3\% \qquad \text{and} \qquad \sigma_k = 0.35 \times \sqrt{1 - 0.7^{2k}}$$

where the volatilties are those for the Black-Karisinsky model specified above; see Section 5.7. That gave the floorlet prices on the right in Figure 14.2, which are *not* discounted. What is plotted is $V_{Fk}/P_{0:k}$ (with $v_0 = 1$) indicating what fraction of the guarantee the option premium is eating up. After a while the cost is down to around 0.15% in all three cases (compared to the guarantee of 3%). In the beginning the expense strongly reflects the state of the economy. For low interest rates the cost must be high since the guarantee is then likely to be used. The reversion to mean effect is again evident in the premia charged.

Options on bonds

The financial markets also offer options on bonds. In the simplest, European case the holder of the option may sell or buy a given bond at an agreed price A at a certain time $T = Kh$. Net pay-offs are similar to options on equity; i.e.

$$\underset{\textit{pay-off put}}{X_P = \max(A - P_{K:K_1}, 0)} \qquad \text{and} \qquad \underset{\textit{pay-off call}}{X_C = \max(P_{K:K_1} - A, 0)}$$

where $P_{K:K1}$ is the value at $T = Kh$ of bonds expiring at $T_1 = K_1 h$. The option holder is in a position to take advantage of the uncertainty in the future price $P_{K:K_1}$.

The entire pricing theory is similar to that for floorlets and caplets. It follows from (1.37) that

$$E_Q(P_{K:K_1}) = P_0(K : K_1); \qquad (1.46)$$

i.e the risk-neutral, expected future bond price conincides with forward contracts traded today. This yields a parity relation between prices V_{PK} and V_{CK} for puts and calls, similar to (1.43). Now

$$V_{PK} - V_{CK} = P_{0:K}\{A - P_0(K : K_1)\}. \qquad (1.47)$$

Pricing is based on a log-normal model for the future bond price $P_{K:K_1}$. Let σ_K be the volatility. Then (1.44) and (1.45) with some of the quantities redefined yields

$$V_{PK} = P_{0:K}\{A\Phi(-a_K + \sigma_K) - P_0(K : K_1)\Phi(-a_K)\} \qquad (1.48)$$

where

$$a_K = \frac{\log(P_0(K : K_1)/A) + \sigma_K^2/2}{\sigma_K}. \qquad (1.49)$$

The price for a call follows from the parity relation.

Interest rate swaps

Swaps are cash flows or other assets two parties may find it mutually beneficial to switch between them. There may be currency swaps, equity swaps, commodity swaps, even volatility swaps; see Hull (2003). Interest rate swaps of the simple **vanilla** form is the exchange of a floating rate r_K for a fixed rate r_S (or the other way around). Many reasons could be advanced for such an agreement to be advantageous for both parties. Swaps have become among the most popular ways of managing risk. The fixed rate r_S is called the **swap rate**.

Consider an interest rate swap lasting from K_1 to K_2. As usual the contract is drawn up at $t_0 = 0$. If v_0 is the capital involved, the amount $(r_S - r_k)v_0$ is exchanged at $t_k = kh$. For the entire scheme the total discounted pay-off is

$$X = \sum_{k=K_1}^{K_2} P_{0:k} \cdot (r_S - r_k)v_0,$$

where the net transfer at t_k may be both positive and negative, depending on whether the floating rate r_k is larger or smaller than the swap rate r_S. The natural price for the contract is the risk-neutral expectation $E_Q(X)$. Now

$$E_Q(X) = \sum_{k=K_1}^{K_2} P_{0:k}\{r_S - E_Q(r_k)\}v_0 = \sum_{k=K_1}^{K_2} P_{0:k}\{r_S - r_0(k)\}v_0,$$

where $r_0(k)$ is the forward interest rate. Contracts are customarily designed so that $E_Q(X) = 0$, involving no premium upfront. The resulting equation can be solved for the swap rate r_S. This yields

$$r_S = \sum_{k=K_1}^{K_2} w_k r_0(k) \qquad \text{where} \qquad w_k = \frac{P_{0:k}}{P_{0:K_1} + \ldots + P_{0:K_2}} \tag{1.50}$$

and the swap rate is a weighted sum of forward rates.

Options on swaps

Another popular interest rate derivative (and the last one to be considered) is options on swaps, also known as **swaptions**. This instrument permits *delayed* entry into a swap on favourable conditions. A **receiver** swaption gives the right to receive from a fixed date $T = Kh$ an agreed, fixed rate r_{SA} in return for the floating rate. There is also the opposite **payer** swaption. Now an agreed, fixed rate r_{SA} is *payed* in exchange for the floating rate. If r_{SK} is the swap rate at $T = Kh$, the option will be exercised if

$$\underbrace{r_{SA} > r_{SK}}_{\textit{receiver swaption}} \qquad \text{or} \qquad \underbrace{r_{SA} < r_{SK}}_{\textit{payer swaption}}$$

i.e. if the agreed conditions are better than the market conditions at the expiry $T = Kh$.

Like all other derivatives a swaptions implies transfer of money if the option holder makes use of the right it implies. The future floating rate is immaterial; only the difference between the floating swap rate r_{SK} and the agreed one r_{SA} counts. At time t_k the responsibility of the option seller is

$$\underbrace{X_{Pk} = \max(r_{SA} - r_{SK})v_0}_{\textit{receiver swaption}} \qquad \text{and} \qquad \underbrace{X_{Ck} = \max(r_{SK} - r_{SA}, 0)v_0,}_{\textit{payer swaption}}$$

v_0 being the capital involved. We are again dealing with a put (receiver) and a call (payer) option, and as above there is a simple parity relation between them (Exercise 14.?). Both

X_{Pk} and X_{Ck} define payment streams starting at $T = Kh$ and extending up to some terminal $T_1 = K_1 h$ of the swap.

A swaption premium is payed up-front at the day ($t_0 = 0$) the deal is struck, and its value hinges on the risk-neutral distribution of the uncertain, future swap rate r_{SK}. Standard pricing is again based on the log-normal model with the mean taken from forward rates observed in the market. In this case these are swap contracts lasting from $T = Kh$ to $T_1 = K_1 h$. If $r_{S0}(K)$ is the forward swap rates for those, it follows from (1.37) that

$$E_Q(r_{SK}) = r_{S0}(K),$$

and all swaption payments can be priced if the volatilitiy σ_{SK} of r_{SK} is known. For the receiver version the formula resemples that of a floor, except that now the cash transferred the swaption holder is the same everywhere; only the discounting varies between time points. It follows that the permium becomes

$$V_{PK} = (P_{0:K_1} + \ldots + P_{0:K_2})\{r_{SA}\Phi(-a_K + \sigma_{SK}) - r_{0S}(K)\Phi(-a_K)\}v_0 \tag{1.51}$$

where

$$a_K = \frac{\log(r_{0S}(K)/r_{SA}) + \sigma_{SK}^2/2}{\sigma_{SK}}. \tag{1.52}$$

Payer swaptions are dicussed in Exercise ??.

1.7 Mathematics of interest rate derivatives

Introduction

Pricing theory for interest rate derivatives is less settled than for equity. One reason is the underlying model. There is for interest rates no default choice the way the geometric random walk is for shares. The simplest approach is to let a *single* stochastic process drive price fluctuations of all money market products. We are then assuming that a European derivative j which expires at T has a value function of the form

$$V_{jk} = V_j(r_k, T - t_k). \tag{1.53}$$

where $\{r_k\}$ a stochastic process (common to all products). We may in practice think of r_k as a floating rate of interest. As it oscillates, so do the values of all money market products, sort of in parallel.

A key issue is again hedging and its impact on valuation. For equity options that viewpoint lead to unique solutions in Section 14.5, but the argument does not carry over, since hedges now are in terms of other derivatives and not the floating rate itself (which isn't an asset that can be bought!). Hedge portfolios must therefore be designed though *other instruments*, and this does not lead to unique prices. It will emerge that we have to be content with a system of consistency requirements that rests on an universal process for the entire money market, the so-called **market price of risk**. Much of the section is an erection of this important concept. For concrete pricing of products it has to be pinned down for concrete pricing of products. The presentation below owes a great deal to Hull (2003).

Technicalities are very much the same as those in Section 14.5. We shall again be dealing with a limiting process where $h \to 0$ and $T = Kh$ is kept fixed. Taylor approximations will be a principal tool. For terms of order at least h two terms in r and one in t is required. The reason is exactly the same as for equity. Remainder terms of size $h^{3/2}$ and smaller can be ignored.

Hedging in the money market

Let $\{V_{1k}\}$ and $\{V_{2k}\}$ be the value processes of a pair of money market derivatives. Their model will be derived from a basic set-up at the end of the section. That will leave us with a joint model of the form $(j = 1, 2)$

$$V_{j,k+1} = V_{jk}(1 + h\zeta_{jk} + \sqrt{h}\,\tau_{jk}\varepsilon_k), \qquad \text{where} \qquad \zeta_{jk} = \zeta_{jk}(r_k), \quad \tau_{jk} = \tau_{jk}(r_k). \quad (1.54)$$

We are allowing drift and volatility to be influenced by $\{r_k\}$. Note that the error process $\{\varepsilon_k\}$ driving the fluctuations are *common to both derivatives.* That is the consequence of the assumption that there is a single stochastic process driving all money market derivatives and products. As usuall the sequence $\varepsilon_1, \varepsilon_2, \ldots$ are independent and Gaussian with mean zero and standard deviation one.

As hedge portfolio consider

$$\mathcal{H}_k = V_{1k} + w_k V_{2k} \qquad \text{with value at } t_{k+1} \qquad \mathcal{H}_{1k} = V_{1,k+1} + w_k V_{2,k+1}.$$

Here w_k is a weight to be determined. Clearly

$$\mathcal{H}_{1k} - \mathcal{H}_k = (V_{1,k+1} + w_k V_{2,k+1}) - (V_{1k} + w_k V_{2k}) = (V_{1,k+1} - V_{1k}) + w_k(V_{2,k+1} - V_{2k}).$$

Inserting (1.54) this becomes

$$\mathcal{H}_{k+1} - \mathcal{H}_k = (V_{1k}\tau_{1k} + w_k V_{2k}\tau_{2k})\sqrt{h}\,\varepsilon_k + (V_{1k}\zeta_{1k} + w_k V_{2k}\zeta_{2k})h$$

after collecting the terms in $\sqrt{h}$ and h. The random term vanishes if we select

$$w_k = -\frac{V_{1k}\tau_{1k}}{V_{2k}\tau_{2k}},$$

and when this weight is used, some simple manipulations show that

$$\mathcal{H}_k = V_{k1}\left(1 - \frac{\tau_{2k}}{\tau_{1k}}\right) \qquad \text{and} \qquad \mathcal{H}_{k+1} - \mathcal{H}_k = V_{1k}\left(\zeta_{1k} - \zeta_{2k}\frac{\tau_{2k}}{\tau_{1k}}\right)h.$$

But since the portfolio is risk-free, these quantities must be related through

$$\mathcal{H}_{k+1} - \mathcal{H}_k = r_k h \mathcal{H}_k,$$

as there would be arbitrage otherwise. Hence

$$V_{1k}\left(\zeta_{1k} - \zeta_{2k}\frac{\tau_{2k}}{\tau_{1k}}\right)h = r_k h V_{1k}\left(1 - \frac{\tau_{2k}}{\tau_{1k}}\right),$$

and with some re-arrangement we have proved that

$$\frac{\zeta_{1k} - r_k}{\tau_{1k}} = \frac{\zeta_{2k} - r_k}{\tau_{2k}}, \qquad (1.55)$$

which holds for *arbitrary* pairs of derivatives in the money market under the assumpitons introduced. What consequences that has, are discussed next.

The market price of risk

The Sharpe ratio (1.55) was encountered in Chapter 5 where it was used to rank investment projects. In the present sequel it defines a universal process $\{\lambda_k\}$ *characterizing the entire money market*. What (1.55) conveys is that the drift and volatility of an *arbitrary* security j in that market may be written

$$\frac{\zeta_{jk}-r_k}{\tau_{jk}}=\lambda_k \qquad \text{or} \qquad \zeta_{jk}=r_k+\tau_{jk}\lambda_k, \tag{1.56}$$

where $\{\lambda_k\}$ is independent of j. This universal process is known as the **market price of risk**, and suggests that prices adjust to a fixed relationship between gain and volatility. As the volatiltiy goes up (increasing risk), then that is compensated by higher expected gain. Or (at least), so is the theory.

Hedging has lead to a **consistency** in the price system of the money market, but the situation differs from the one we met for equities where the hedging/no arbitrage argument lead to a specific price. Here the valuation of interest rate derivatives demands a concrete position on the market price of risk.

The pricing formula

The martingale pricing used in Section 14.6 has a simple link to the preceding theory. Consider $M_k=V_{1k}/V_{2k}$ where V_{1k} and V_{2k} are the value processes introduced in (1.54). Now

$$M_{k+1}=\frac{V_{1,k+1}}{V_{2,k+1}}=M_k\frac{1+h\zeta_{1k}h+\tau_{1k}\sqrt{h}\,\varepsilon_k}{1+h\zeta_{2k}h+\tau_{2k}\sqrt{h}\,\varepsilon_k}.$$

after inserting the defining model (1.54). We shall utilize that for small x is

$$\frac{1}{1+x}\doteq 1-x+x^2.$$

which is applied to the denominator in the relationship for M_{k+1}. Take $x=h\zeta_{2k}h+\tau_{2k}\sqrt{h}\,\varepsilon_k$. Then

$$M_{k+1}\doteq M_k(1+h\zeta_{1k}+\tau_{1k}\sqrt{h}\,\varepsilon_k)\{1-(h\zeta_{2k}+\tau_{2k}\sqrt{h}\,\varepsilon_k)+(\zeta_{2k}h+\tau_{2k}\sqrt{h}\,\varepsilon_k)^2\}.$$

When the right hand side is muliplied out and the terms of order $\sqrt{h}$ and h are collected, those of order $\sqrt{h}$ cancel (!), and we are left with

$$M_{k+1}\doteq M_k\{1+h(\zeta_{1k}-\zeta_{2k}-\tau_{1k}\tau_{2k}+\tau_{2k}^2\varepsilon_k^2)\}$$

Taking expectations

$$E(M_{k+1})\doteq E(M_k)\{1+h(\zeta_{1k}-\zeta_{2k}-\tau_{1k}\tau_{2k}+\tau_{2k}^2)\}. \tag{1.57}$$

The error is of order x^3 or $h^{3/2}$, small enough for it to go away as $h\to 0$ and $K=T/h\to\infty$, as explained in the introduction to Section 14.5.

Suppose we make the assumption that the market price of risk coincides with the volatility of the *second* uncertain quantity, admittedly rather arbitrary. This means that $\lambda_k = \tau_{2k}$ and that

$$\zeta_{1k} - \zeta_{2k} = \lambda_k(\tau_{1k} - \tau_{2k}) = \tau_{2k}(\tau_{1k} - \tau_{2k}) = \tau_{1k}\tau_{2k} - \tau_{2k}^2$$

so that (1.57) becomes $E(M_{k+1}) \doteq E(M_k)$, implying that $E(M_K) \doteq E(M_0)$. In the limit as $h \to 0$, we obtain the pricing equation

$$E_Q(M_K) = E_Q(M_0), \tag{1.58}$$

after introducing risk neutrality in the notation. This leads directly to (1.36) when $M_k = V_k/P_{k:K}$ is inserted ($P_{k:K}$ is the bond price).

Modelling framework

It remains to derive the model (1.54) on which the pricing theory rests. We shall start from the general model

$$r_{k+1} = r_k e^{h\xi_k + \sqrt{h}\,\sigma_k\varepsilon_k} \qquad \text{where} \qquad \xi_k = \xi_k(r_k), \;\; \sigma_k = \sigma_k(r_k), \tag{1.59}$$

actually the framwork used with equity in Section 14.5. There are time-varying drift and volatility functions (depending on the current floating rate), and (as usual) the process $\{\varepsilon_k\}$ is independent $N(0,1)$. Of course concrete specifications and parameters would now be different (reversion to mean for example), but in the present context this does not matter. The exponential form was used with the Black-Karisinsky model in Section 5.7 (and elsewhere), but we could equally well have started from the linear version (as in the Vasiček model); arguments are virtually unchanged. Volatilities proportional to $\sqrt{h}$ are also as for equities and were for interest rates justified in Section 5.7 in a special case.

Let $V_k = V(r_k, T - t_k)$ be the value of some derivative at t_k. As with equities we are assuming that $V(r,t)$ is a smooth function that can be studied through a Taylor series. Indeed, the following steps are exactly those carried out in Section 14.4 and 14.5 (but spread out a bit there). Start by noting that

$$V(r,t) \doteq V(r_k,t_k) + a_k(r - r_k) + \frac{1}{2}b_k(r - r_k)^2 + c_k(t - t_k)$$

where

$$a_k = \frac{\partial V(r_k,t_k)}{\partial r}, \qquad c_k = \frac{\partial^2 V(r_k,t_k)}{\partial r^2}, \qquad b_k = \frac{\partial V(r_k,t_k)}{\partial t}.$$

Take $r = r_{k+1}$ and $t = t_{k+1}$, which yields

$$V_{k+1} - V_k \doteq a_k(r_{k+1} - r_k) + \frac{1}{2}b_k(r_{k+1} - r_k)^2 + c_k(t_{k+1} - t_k).$$

The quantities $r_{k+1} - r_k$ can be simplified in the same manner as for equity prices. Indeed, recall the Taylor series of the exponential function, the first three terms of which being

$$e^x \doteq 1 + x + \frac{1}{2}x^2.$$

Take $x = \xi_k h + \sigma_k \sqrt{h}\,\varepsilon_k$ and deduce that

$$e^{\xi_k h + \sigma_k \sqrt{h}\,\varepsilon_k} \doteq 1 + \xi_k h + \sigma_k \sqrt{h}\,\varepsilon_k + \frac{1}{2}(\xi_k h + \sigma_k \sqrt{h}\,\varepsilon_k)^2 \doteq 1 + \sqrt{h}\sigma_k\varepsilon_k + h(\xi + \frac{1}{2}\sigma_k^2\varepsilon_k^2),$$

ignoring terms of terms of order $h^{3/2}$ and smaller. It follows that

$$r_{k+1} - r_k = (e^{\xi_k h + \sigma_k \sqrt{h}\,\varepsilon_k} - 1)r_k \doteq \sqrt{h}\sigma_k\varepsilon_k + h(\xi + \frac{1}{2}\sigma_k^2\varepsilon_k^2)r_k.$$

or, using the auxiliary result at the end of Section 14.5

$$r_{k+1} - r_k \doteq \sqrt{h}\sigma_k\varepsilon_k + h(\xi + \frac{1}{2}\sigma_k^2)r_k.$$

This may be inserted into the Taylor series for V_k. Then after collecting terms up to order h it follows that

$$V_{k+1} = V_k(1 + h\zeta_k + \sqrt{h}\,\tau_k\varepsilon_k) \tag{1.60}$$

where

$$\zeta_k = a_k\zeta_k + \frac{1}{2}b_k\sigma_k^2 + c_k \qquad \text{and} \qquad \tau_k = a_k\sigma_k,$$

as assumed in (1.54). Note that under the assumption (1.53) the random terms ε_k are the same for all derivatives whereas the functions ζ_k and τ_k will vary from one product to another.

1.8 Further reading

1.9 Exercises

Exercise 1
Suppose we form a portfolio consisting of the three instruments in the table:

Instrument	Value at $t_0 = 0$
European call	$C_0(S_0)$
- Underlying asset	$-S_0$
Cash	$\exp(-rT)A$

Note that we are long in the option and short in the underlying stock.

a) Determine the value of the three ingrdients of the table at the time of expiry of the option and show that the value of the whole portfolio then is

$$\max(A - S_K, 0),$$

b) Use this arbitrage and this result to deduce the lower bound (**??**) on European call options.

Exercise 2
Construct a portfolio similar to that in Exercise 1 that enables you to deduce the lower bound (**??**) for European put options.