Exercise 16

A cap is a derivative where the buyer is reimbursed if the interest on some loan of size L exceeds the so-called cap rate R_c. Suppose the agreement is going to apply from t_0 to $T = t_0 + Kh$ and that pay-offs may take place at $t_k = t_0 + kh$, $k = 1, 2, \ldots, K$. There is no payment at the start. Compensations always take place *after* the event, based on the spot rate R_k of the preceding period. The actual pay-off

$$Lh \max(R_k - R_c, 0)$$

is then received at time t_{k+1} by the buyer of the cap. This is known as a *caplet*. In practice the agreement may well be deferred so that it starts running from some future time t_0.

a) Discount the pay-off to apply at time t_k (by dividing on $1 + R_k h$) and demonstrate that you then get

$$\max(L - \frac{L(1 + R_c h)}{1 + R_k h}, 0).$$

Interprete this as a put option on a zero-coupon bond.

Let $P(0, t)$ be the price of such a zero-coupon bond maturing at time t; i.e. the price you pay now for receiving one money unit at time t. This quantity goes into the standard valuation of the cap.

Another assumption is that the spot rate R_k is log-normal with volatility σ_k. Then the present value of the caplet is customary set to

$$LhP(0, t_{k+1})\{F_k \Phi(d_1) - R_c \Phi(d_2)\}$$

where Φ is the standard normal integral, F_k the forward rate of interest for the period from t_k to t_{k+1} and

$$d_1 = \frac{\log(F_k/R_c) + \sigma_k^2 t_k/2}{\sigma_k \sqrt{t_k}}, \qquad d_2 = d_1 - \sigma_k \sqrt{t_k}$$

Note that the forward spot rate F_k is a quanitity you observe in the market today, as the deal is struck.

The bond price is calculated with an annual interest rate of 0.06.

b) Find a formula for the bond price

c) Write a program that computes the present value of a cap.

Suppose that

$$L = 10000, \quad F_k = 0.07, \quad R_c = 0.08, \quad h = 0.25, \quad \sigma_k = 0.20, \quad \sigma_k = 0.20, \quad t_0 = 1 \quad , K = 20$$

d) Describe what the cap agreement amounts to in practical terms.

e) What should the premium for the total cap be? Find out how much it varies with

the future spot rate and the volatility.

f) Write a program which simulates the actual pay-off under the cap.

g) Carry out simulations of the cap pay-off where the model for the logarithm of the spot rate R_k is autoregressive of order 1. Ensure that mean spot rate is 0.065 and use $a = 0.65$ as autoregressive parameter.