Exercise 4.1

The swap rate r_0 for a swap between $t_J = Jh$ and $t_L = Lh$ is defined as

$$s_0 = \sum_{k=J+1}^{L} w_k r_0(k) \qquad \text{where} \qquad w_k = \frac{P(0{:}k)}{P(0{:}J{+}1) + \ldots + P(0{:}L)}$$

where $r_0(k)$ is the forward rate of interest between t_{k-1} and t_k.

a) Dig out the relationship bewteen forward rates of interest and bond prices from somewhere and prove that the swap rate can be re-written

$$s_0 = \frac{P(0{:}J) - P(0{:}L)}{P(0{:}J{+}1) + \ldots + P(0{:}L)}.$$

Suppose we have an built an analytical model for bond pricing from somewhere with $P(r_0, T)$ being the value of a bond expiring at T when r_0 is the rate of interest today.

b) Argue that a reasonable model for the future swap rate s_k at t_k is

$$s_k = \frac{P(r_k, J) - P(r_k, L)}{P(r_k, J{+}1) + \ldots + P(r_k, L)}$$

where it is assumed that the underlying swap is between an interval that is shifted with the movement of k.

c) Develope an alternativ model with the interval fixed at t_J and t_L regardless of k (now we must assume $k < J$).

We shall consider *simulated* swap rates.

d) Try to discuss whether the model for r_k then should be the same as the one underlying the bond price model.

e) Develope simulation programs for both c) and d) when $P(r_0, T)$ is given by the Vasicěk term structure

$$P(r_0, T) = e^{A(T) - B(T) r_0}$$

where

$$B(T) = \frac{1 - e^{-a_0 t}}{a_0} \qquad \text{and} \qquad A(T) = (B(T) - T)\left(\xi - \frac{\sigma_0^2}{2a_0^2}\right) - \frac{\sigma_0^2 B(T)^2}{4a_0}$$

and r_k is follows the Black-Karisinsky model.

Suppose $J = 5$ and $L = 10$ years and use the parameter setting in Exercise 3.2.f for the Vasicěk bond model and $\xi = 4\%$, $\sigma = 0.25$ and $a = 0.7$ for the Black-Karisinsky model.

f) Simulate s_k 20 years ahead for the case in b) and 5 years for that in c). Start at $r_0 = 2\%$, repeat 20 times and plot the rates together. Any comments?