Introducing copulas

Introduction

Let U_1 and U_2 be uniform, *dependent* random variables and introduce

$$X_1 = F_1^{-1}(U_1) \qquad \text{and} \qquad X_2 = F_2^{-1}(U_2), \tag{0.1}$$

where $F_1^{-1}(u_1)$ and $F_2^{-1}(u_2)$ are the percentiles of two distribution functions $F_1(x)$ and $F_2(x)$. This simple set-up defines an increasingly popular modelling strategy where dependence and univariate variation are treated separately. The inversion algorithm (Section 2.3) ensures that the distribution functions of X_1 and X_2 become $F_1(x_1)$ and $F_2(x_2)$ no matter how U_1 and U_2 depend on each other. Their joint distribution function $C(u_1, u_2)$ is called a **copula** and a huge number of models have been proposed for it.

The idea goes back to the mid twentieth century, originating with the work of Sklar (1959). It enables us to tackle situations as those in Figure 6.5 where the correlation depend on the level of the variables. Equity in falling markets is an example (Longin and Solnik 2001), and such phenomena have drawn interest in insurance too; see Wütrich (2004).

Copula modelling

A bivariate copula is the joint distribution function

$$C(u_1, u_2) = \Pr(U_1 \le u_1, U_2 \le u_2), \qquad 0 < u_1 \le 1,\ 0 < u_2 \le 1. \tag{0.2}$$

Any function $C(u_1, u_2)$ that is to play this role must be increasing in u_1 and u_2 and satisfy

$$C(u_1, 0) = 0,\ \ C(0, u_2) = 0 \qquad \text{and} \qquad C(u_1, 1) = u_1,\ \ C(1, u_2) = u_2 \tag{0.3}$$

where the conditions on the right ensures that U_1 and U_2 are uniform. Simple examples are

$$\underbrace{C(u_1, u_2) = u_1 u_2}_{\textit{independent copula}} \qquad \text{and} \qquad \underbrace{C(u_1, u_2) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{-1/\theta}}_{\textit{Clayton copula}}$$

where $\theta > 0$ on the right. You can easily convince yourself that (0.3) is valid for both.

The copula approach rests on a representation theorem discovered by Sklar (1959). Any joint distribution function $F(x_1, x_2)$ with strictly increasing marginal distribution functions $F_1(x_1)$ and $F_2(x_2)$ may be written

$$\underbrace{F(x_1, x_2) = C(u_1, u_2)}_{\textit{copula modelling}} \qquad \text{where} \qquad \underbrace{u_1 = F_1(x_1),\ \ u_2 = F_2(x_2)}_{\textit{univariate modelling}} \tag{0.4}$$

with a modified version even for counts. The copula approach does not restrict the model at all, and there are additional versions when antitetic twins (Section 4.3) are supplied for the uniforms. Indeed, the copula on the left in (0.4) may be combined with either of

$$\begin{array}{ll} u_1 = F_1(x_1),\ \ 1 - u_2 = F_2(x_2) & \textit{orientation (1,2)} \\ 1 - u_1 = F_1(x_1),\ \ u_2 = F_2(x_2) & \textit{orientation (2,1)} \\ 1 - u_1 = F_1(x_1),\ \ 1 - u_2 = F_2(x_2) & \textit{orientation (2,2),} \end{array} \tag{0.5}$$

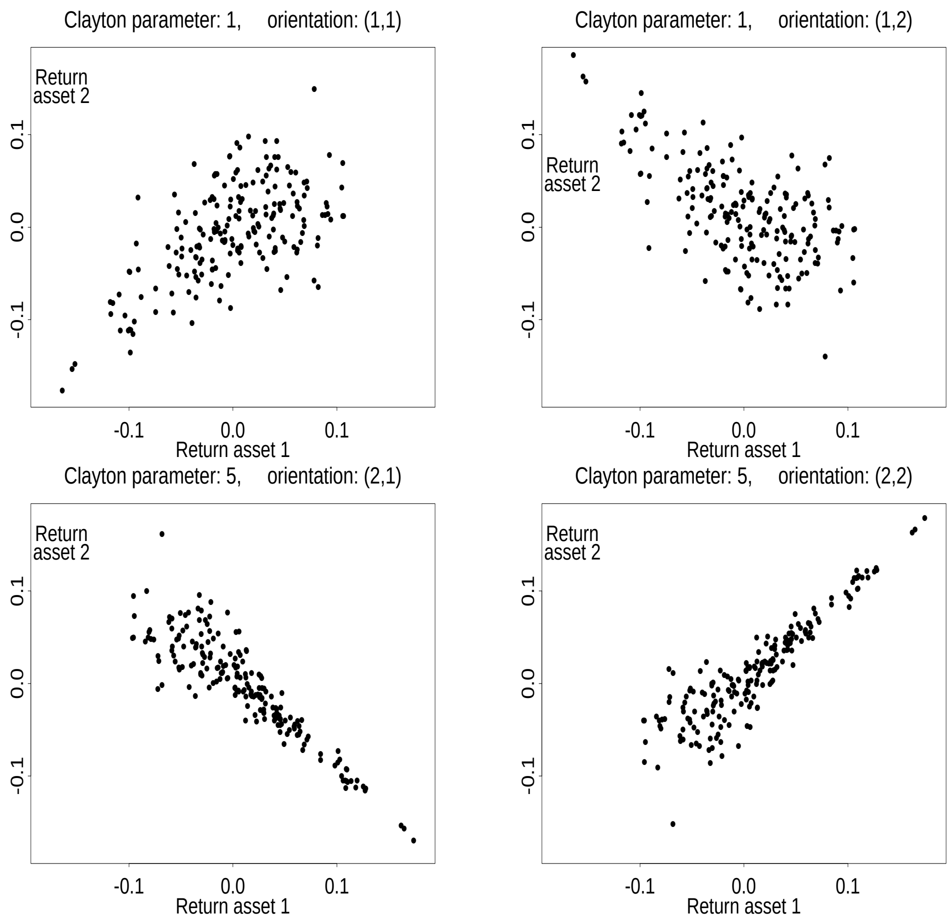

Figure 6.5 *Simulated financial returns for Gaussians combined with the Clayton copula.*

and the effect is to rotate the copula patterns 90°, 180° and 270° compared to the orginal one denoted orientation $(1,1)$; see Figure 6.5.

The Clayton copula

The definition of the Clayton copula can be extended to

$$C(u_1,u_2) = \max\left((u_1^{-\theta} + u_2^{-\theta} - 1)^{-1/\theta}, 0\right), \qquad \theta \geq -1 \tag{0.6}$$

where it is easy to check that the copula requirements (0.3) are satisfied for all $\theta \neq 0$ (and ≥ -1). Nor is it difficult to show (Exercise 6.7.3) that $C(u_1,u_2) \to u_1u_2$ as $\theta \to 0$ which means that $\theta = 0$ is the independent case. When θ is negative,

$$C(u_1,u_2) = 0 \qquad \text{if} \qquad u_2 < (1 - u_1^{-\theta})^{-1/\theta},$$

and certain pairs (u_1,u_2) are forbidden territory. Hard restrictions of that kind are often undesirable. Yet when negative θ is included, the family in a sense covers the entire range of dependency that is logically possible.; see Exercises 6.7.1.and 6.7.2.

Simulated structures under the Clayton copula are shown in Figure 6.5 for normal X_1 and X_2 with mean $\xi = 0.005$ and volatility $\sigma = 0.05$, precisely as in Figure 2.5 (and realistic for monthly equity returns). The cone-shapes patterns signify unequal dependence in unequal parts of the space. Note, for example, the plot in the upper, left corner where correlations are much stronger for downside returns. Ordinary Gaussian models don't capture such phenomena which do appear in real life; see Longin and Solnik (2001). The other plots in Figure 6.5 rotate patterns by changing the orientation of the copula (two of them have become *negatively* correlated), and the degree of dependence is adjusted by varying θ.

Conditional distributions under copulas

Additional insight is gained by examining conditional distributions. The conditional and joint density functions are related to $C(u_1, u_2)$ through

$$c(u_2|u_1) = c(u_1, u_2) = \frac{\partial^2 C(u_1, u_2)}{\partial u_1 u_2},$$

and when this is integrated with respect to u_2

$$C(u_2|u_1) = \int_0^{u_2} c(v|u_1)\, dv = \int_0^{u_2} \frac{\partial^2 C(u_1, v)}{\partial u_1 \partial v}\, dv = \frac{\partial}{\partial u_1} \int_0^{u_2} \frac{\partial C(u_1, v)}{\partial v}\, dv = \frac{\partial C(u_1,u_2)}{\partial u_1}.$$

For the Clayton copula (0.6)

$$C(u_2|u_1) = u_1^{-(1+\theta)} \max\left((u_1^{-\theta} + u_2^{-\theta} - 1)^{-(1+1/\theta)}, 0 \right), \tag{0.7}$$

where the expression is zero when $\theta < 0$ and $u_2 < (1 - u_1^{-\theta})^{-1/\theta}$. It has been plotted in Figure 6.6 with θ large and positive on the left and large and negative on the right. Shapes under $u_1 = 0.1$ and $u_2 = 0.9$ differ markedly, a sign of strong dependence, but the most notable feature is a lack of symmetry. On the left U_2 is largely confined to a narrow strip around u_1 when $u_1 = 0.1$, but is much more variable when $u_1 = 0.9$. It is precisely this feature that creates the cones in Figure 6.5.

The preceding argument may be flawed when the **support** of (U_1, U_2) (i.e. the region of positive probability) doesn't cover the entire unit quadrate. Clayton copulas with negative θ is such a case though for those the argument does go through; see Genest and Mackay (1986) for a simple account of these issues.

Many variables and the Archimedean class

Copulas can be extended to any number of variables. A J-dimensional one is the distribution function $C(u_1, \ldots, u_J)$ of J dependent, uniform variables $U_1, \ldots, U_J$ and satisfies consistency requirements similar to those in (0.3); see Exercise 6.7.6. Transformations back to the original variables are now through $X_1 = F_1^{-1}(U_1), \ldots, X_J = F_J^{-1}(U_J)$, and there are 2^J ways of rotating patterns through antitetic twins, not just 4. Copulas of sub-vectors follows from higher-order ones. For example, if $j < J$ then, $C(u_1, \ldots, u_j, 1, \ldots, 1)$ is the copula for $U_1, \ldots, U_j$.

Arguably the most convenient J-dimensional copulas are the **Archimedean** ones

$$C(u_1, \ldots, u_J) = \phi^{-1}\{\phi(u_1) + \ldots + \phi(u_J)\} \tag{0.8}$$

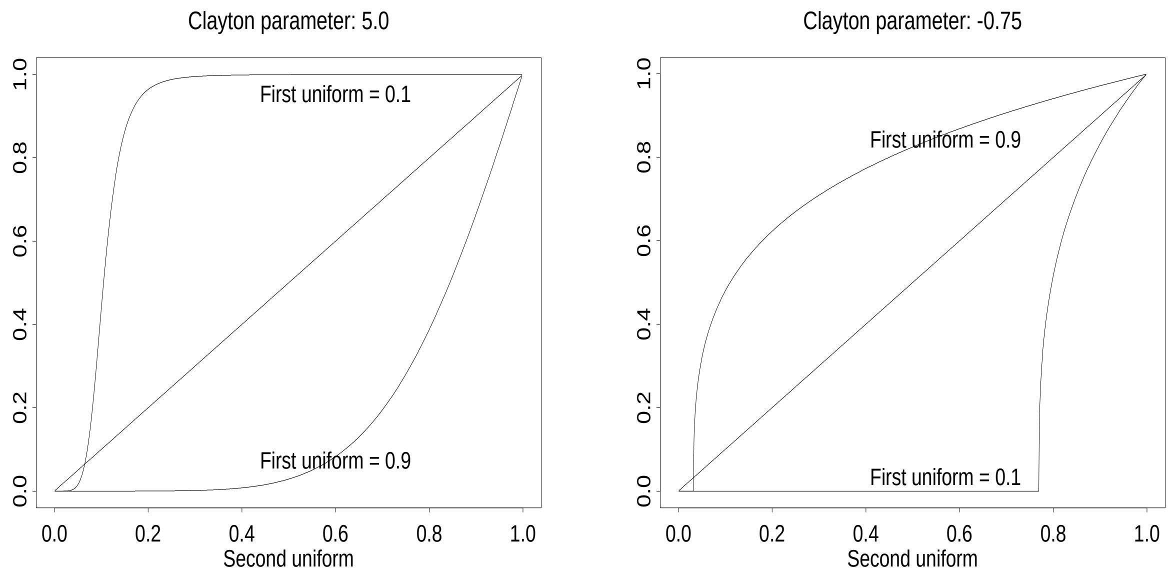

Figure 6.6 *Conditional distribution functions for the* **second** *variable of a Clayton copula; given first variable marked on each curve.*

where $\phi(u)$ with inverse $\phi^{-1}(x)$ is known as the **generator**. The Clayton copula is the special case

$$\phi(u) = \frac{1}{\theta}(u^{-\theta} - 1), \qquad \text{and} \qquad \phi^{-1}(x) = (1 + \theta x)^{-1/\theta}$$

from which (0.6) follows. It is usual to let $\theta \geq 0$ so that the support is the entire J-dimensional unit quadrate. A huge list of generators is compiled in Nelsen (2006). They must in general be monotone functions with $\phi(1) = 0$, and it is usually desirable that $\phi(0) = \infty$. If not, certain sub-regions of the unit quadrate are completely ruled out which we may not want. Thus, generators for practical use are more likely to look like the Clayton copula for $\theta = 0.2$ on the right of Figure 6.7 than the second example $\phi(u) = (1 - u)^3$ on the right. Archimedean copulas go back to Kimberling (1974).

The Marshall-Olkin representation

Some of the most useful Archimedean copulas satisfy a stochastic representation due to Marshall and Olkin (1988). Let Z be a positive random variable with density function $g(z)$. Its moment generating function (or **Laplace transform**) is

$$M(x) = E(e^{-xZ}) = \int_0^\infty e^{-xz} g(z)\, dz; \tag{0.9}$$

see Section A.1. Only positive x is of interest, and $M(x)$ decreases monotonely from one at $x = 0$ to zero at infinity. Define

$$U_j = M\left(-\frac{\log(V_j)}{Z}\right), \qquad j = 1, \ldots, J \tag{0.10}$$

where $V_1, \ldots, V_J$ is a sequence of independent and uniform random variables. It turns out that $U_1, \ldots, U_J$ are uniform too (not exactly obvious!), and their joint distribution function is an Archimedian copula with generator $\phi(u) = M^{-1}(u)$; see Section 6.8 for the proof.

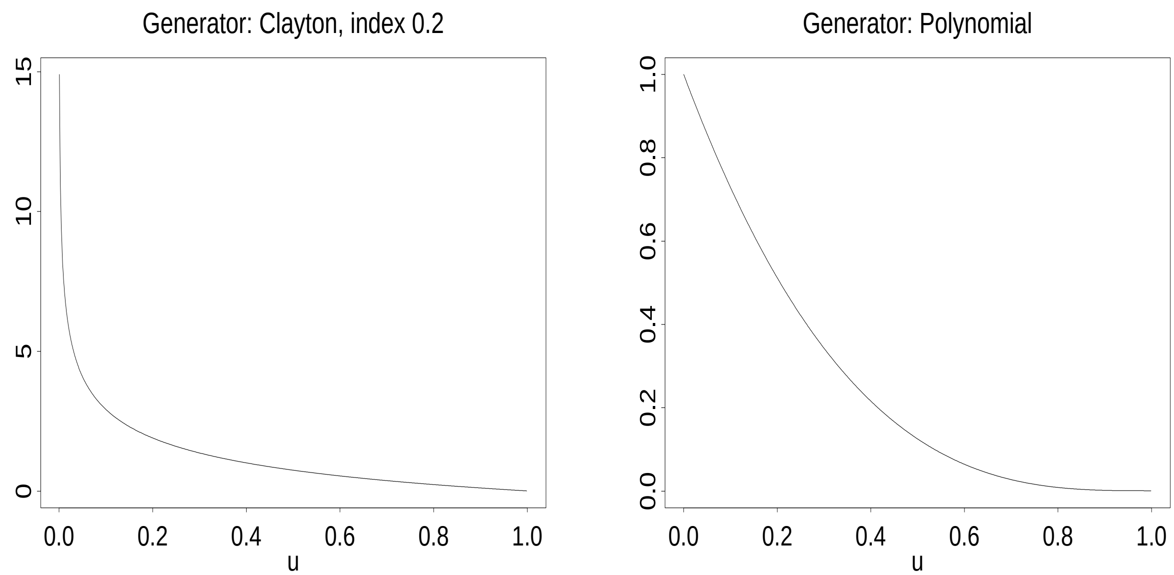

Figure 6.7 *Generator functions for Archimedean copulas.*

The Clayton copula emerges when Z is Gamma distributed with density function

$$g(z) = \frac{\alpha^\alpha}{\Gamma(\alpha)} z^{\alpha-1} e^{-\alpha z} \qquad \text{where} \qquad \alpha = 1/\theta.$$

Then

$$M(x) = \int_0^\infty e^{-xz} \frac{\alpha^\alpha}{\Gamma(\alpha)} z^{\alpha-1} e^{-\alpha z}\, dz = \left(1 + \frac{x}{\alpha}\right)^{-\alpha} = (1+\theta x)^{-1/\theta}$$

which is the inverse Clayton generator. Positive distributions with simple moment generating functions and inverses are natural candidates for Z.

Copula sampling

A Monte Carlo simulation $U_1^*, \ldots, U_J^*$ of a copula vector is passed on to the original variables through

$$X_1^* \leftarrow F_1^{-1}(U_1^*), \ldots, X_J^* \leftarrow F_J^{-1}(U_J^*).$$

This is inversion sampling which does not work for all distributions, but the table look-up algorithm of Section 4.2 (which is an *approximate* inversion) is a satisfactory way around.

What about $U_1^*, \ldots, U_J^*$ itself? A general recursive scheme will be worked out in Section 6.8, but it is far from being universally practical. One class of models that *is* easy to handle are Archimedean copulas under the Marshall-Olkin stochastic representation. When (0.10) is copied in the computer:

Algorithm 6.3 Archimedean copulas

0 Input: $\phi(u)$

Draw Z^* $\qquad$ *%Z with Laplace transform* $M(u) = \phi^{-1}(u)$
For $j = 1, \ldots, J$ repeat
$\qquad$ Draw $V^* \sim$ uniform $\quad$ and $\quad U_j^* \leftarrow M(-\log(V^*)/Z^*)$
Return $U_1^*, \ldots, U_J^*$

A simulation from the Clayton copula is generated if Z^* is drawn from the standard Gamma distribution with shape $\alpha = 1/\theta$. There are for Clayton copulas an alternative which is justified by the scheme in Section 6.8:

Algorithm 6.4 The Clayton copula
Input: $\theta > 0$
Draw $U_1^* \sim$ uniform $\quad$ and $\quad S^* \leftarrow 0$
For $j = 2, \ldots, J$ do
$\qquad S^* \leftarrow S^* + (U_{j-1}^*)^{-\theta} - 1$ $\qquad$ *%Updating from preceding uniform*
$\qquad$ Draw $V^* \sim$ uniform
$\qquad U_j^* \leftarrow \{(1 + S^*)(V^*)^{-\theta/(1+(j-1)\theta)} - S^*\}^{-1/\theta}$ $\qquad$ *%Next uniform*
Return $U_1^*, \ldots, U_J^*$

The sample $U_1^*, \ldots, U_J^*$ emerging from this second algorithm are smooth functions of θ. Why is this useful? Consider the following:

Example: An equity portfolio
Let $\mathcal{R} = (R_1 + R_2)/2$ be the return of an equally weighted portfolio with individual assets yielding

$$R_1 = e^{\xi_1 + \sigma_1 \varepsilon_1} - 1 \qquad \text{and} \qquad R_2 = e^{\xi_2 + \sigma_2 \varepsilon_2} - 1$$

where ε_1 and ε_2 are both $N(0, 1)$. Suppose they are Clayton-dependent with parameter θ. The lower 5% percentile and the standard deviation of the portfolio are in Figure 6.8 plotted against its inverse $\alpha = 1/\theta$ when $\xi_1 = \xi_2 = 0.005$ and $\sigma_1 = \sigma_2 = 0.05$. This might be monthly returns for equity. Both downside and variability depend sensitively on α with low α for strong dependency between asset returns.

There is a more technical side to the display in Figure 6.8. Monte Carlo ($m = 10000$ simulations) was used for computation with common random numbers (Section 4.3) to smooth the curves. The random number generator was then run from the same start for each of the 100 values of α plotted; see Section 4.6 where the issue is explained. Yet the picture is smooth *only* when the copulas were sampled by Algorithm 6.4. Why the erratic behaviour when Algorithm 6.3 was used? The reason is the underlying Gamma variables being generated by Algorithm 2.9 which is a random stopping rule with which common random numbers does *not* work well.

Example: Copula log-normals against pure log-normals
If the preceding copula log-normal is indeed the true model, how wrong is it to use the traditional log-normal instead? Comparisons of that kind require careful calibration of models. The univariate part is defined by $\varepsilon_1 = \Phi^{-1}(U_1)$ and $\varepsilon_2 = \Phi^{-1}(U_2)$ where $\Phi^{-1}(u)$ is Gaussian percentiles. Now $\rho = \text{cor}(\varepsilon_1, \varepsilon_2)$ is by definition

$$\rho = E\{\Phi^{-1}(U_1), \Phi^{-1}(U_2)\},$$

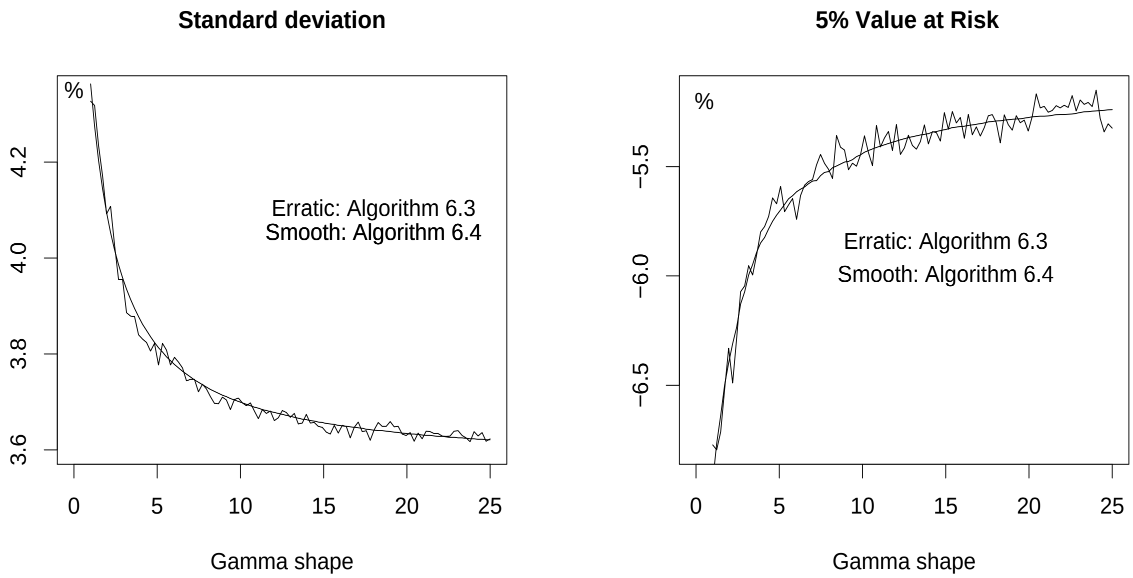

Figure 6.8 *Standard deviation (**left**) and lower* 5% *percentiles (**right**) under variation of the inverse Clayton parameter for the equity portfolio described in the text ($m = 10000$ simulations).*

and its value should be the same whether (U_1, U_2) comes from the Clayton copula or corresponds to an ordinary bivariate normal.

The experiments reported in Figure 6.9 were based on the Clayton copula with $\theta = 1$, and the corresponding ρ for the pure Gaussian was determined by Monte Carlo. If (U_{1i}^*, U_{2i}^*) for $i = 1, \dots, m$ are simulations of (U_1, U_2) under the copula, then ρ is approximated by

$$\rho^* = \frac{1}{m} \sum_{i=1}^{m} \Phi^{-1}(U_{1i}^*) \Phi^{-1}(U_{2i}^*)$$

which gave $\rho^* = 0.498$. The other conditions were those of the preceding example. Density functions of $\mathcal{R}$ are plottet in Figure 6.9 (one million simulations used). To the left all .simulations exceeding -5% were discarded, and the rest used to portray the extreme downside. Discrepancies under the two models are not negligable, but this changes on the right of Figure 6.9 which shows density functions of five-year returns. Differences are now hardly visible at all. Five-year (or sixty-month) returns were simulated through the recursion

$$\mathcal{R}_{0:k}^* = \mathcal{R}_{0:k-1}^*(1 + \mathcal{R}_k^*), \quad k = 1, \dots, 60 \qquad \text{starting at} \qquad \mathcal{R}_{0:0}^* = 1$$

where $\mathcal{R}_k^*$ is the Monte Carlo return in period k. Long-range asset risk may not be too strongly influenced by subtle copula effects. Other values of θ gave similar results.

Mathematical arguments

The Marshall-Olkin representation Let $V_1, \dots, V_J, Z$ be independent random variables with $V_1, \dots, V_J$ uniform and Z positive with density function $g(z)$ and moment generating function

$M(x) = \int_0^\infty e^{-xz} g(z) dz$. Define $U_j = M(-\log(V_j)/Z)$ for $j = 1, \ldots, J$. We shall prove that $U_1, \ldots, U_J$ follow an Archimedean copula with generator $\phi(u) = M^{-1}(u)$. First note that $V_j = e^{-M^{-1}(U_j)Z}$. Hence, if $Z = z$ is fixed, then

$$\Pr(U_1 \leq u_1, \ldots, U_J \leq u_J | z) = \Pr(V_1 \leq e^{-M^{-1}(u_1)z} \ldots, V_J \leq e^{-M^{-1}(u_J)z} | z)$$

and since $V_1, \ldots, V_J$ are independent and uniform, this yields

$$\Pr(U_1 \leq u_1, \ldots, U_J \leq u_J | z) = e^{-M^{-1}(u_1)z - \ldots - M^{-1}(u_J)z}.$$

But

$$\Pr(U_1 \leq u_1, \ldots, U_J \leq u_J) = \int_0^\infty \Pr(U_1 \leq u_1, \ldots, U_J \leq u_J | z) g(z)\, dz$$

so that

$$\Pr(U_1 \leq u_1, \ldots, U_J \leq u_J) = \int_0^\infty e^{-\{M^{-1}(u_1) + \ldots + M^{-1}(u_J)\}z} g(z)\, dz$$

which can also be written

$$\Pr(U_1 \leq u_1, \ldots, U_J \leq u_J) = M\{M^{-1}(u_1) + \ldots + M^{-1}(u_J)\}.$$

This a Archimedean copula with generator $\phi(u) = M^{-1}(u)$.

A general scheme for copula sampling

Some copulas can be sampled through inversion. Start by drawing J uniforms U_1^* and $V_2^*, \ldots, V_J^*$ and proceed iteratively through

$$C(U_j^* | U_1^*, \ldots, U_{j-1}^*) = V_j^*, \qquad j = 2, \ldots, J \tag{0.11}$$

where $C(u_j | u_1, \ldots, u_{j-1})$ is the conditional distribution function of U_j given $U_1 \ldots, U_{j-1}$. This would yield the desired vector $U_1^*, \ldots, U_J^*$, but whether it is practical hinges on the work needed to solve the equations. We must in any case derive an expression for $C(u_j | u_1, \ldots, u_{j-1})$. Let $c(u_j | u_1, \ldots, u_{J-1})$ be its density function so that

$$C(u_j | u_1, \ldots, u_{j-1}) = \int_0^{u_j} c(v | u_1, \ldots, u_{j-1})\, dv.$$

To calculate the integral let $c(u_1, \ldots, u_j)$ be the density function of $U_1, \ldots, U_j$ and recall that $c(u_1, \ldots, u_{j-1}, 1)$ is the density functions $U_1, \ldots, U_{j-1}$. It follows that

$$c(u_j | u_1, \ldots, u_{j-1}) = \frac{c(u_1, \ldots, u_j)}{c(u_1, \ldots, u_{j-1}, 1)} = \frac{\partial^j C(u_1, \ldots, u_j)/\partial u_1, \ldots, \partial u_j}{\partial^{j-1} C(u_1, \ldots, u_{j-1}, 1)/\partial u_1, \ldots, \partial u_{j-1}}.$$

Write D for the denominator. Then

$$\begin{aligned}\int_0^{u_j} c(v | u_1, \ldots, u_{j-1})\, dv &= D^{-1} \int_0^{u_j} \frac{\partial^j C(u_1, \ldots, u_{j-1}, v)}{\partial u_1 \ldots, \partial u_{j-1} \partial v}\, dv \\ &= D^{-1} \frac{\partial^{j-1}}{\partial u_1, \ldots, \partial u_{j-1}} \int_0^{u_j} \frac{\partial C(u_1, \ldots, u_{j-1}, v)}{\partial v}\, dv \\ &= D^{-1} \frac{\partial^{j-1} C(u_1, \ldots, u_j)}{\partial u_1 \ldots \partial u_{j-1}},\end{aligned}$$

and it has been established that

$$C(u_j|u_1,\ldots,u_{j-1})=\frac{\partial^{j-1}C(u_1,\ldots,u_{j-1},u_j)/\partial u_1\ldots\partial u_{j-1}}{\partial^{j-1}C(u_1,\ldots,u_{j-1},1)/\partial u_1\ldots\partial u_{j-1}} \tag{0.12}$$

which is the extension of the bivariate case in Section 6.7. If the equation (0.11) is easy to solve after having calculated the derivative, we have a sampling method.

Justifying Algorithm 6.4

The preceding recursive technique works for Clayton copulas if $\theta>0$. Then

$$C(u_1,\ldots,u_j)=\left(\sum_{i=1}^{j}u_i^{-\theta}-j+1\right)^{-1/\theta},$$

which is easily be differentiated with respect to $u_1,\ldots,u_{j-1}$. After a little work

$$\frac{\partial^{j-1}C(u_1,\ldots,u_j)}{\partial u_1\ldots\partial u_{j-1}}=\left(\sum_{i=1}^{j}u_i^{-\theta}-j+1\right)^{-1/\theta-j+1}\times\prod_{i=1}^{j-1}\left(u_i^{-(1+\theta)}\{1+(i-1)\theta\}\right),$$

and the conditional distribution function of U_j given $u_1,\ldots,u_{j-1}$ may now be calculated from (0.12). This yields

$$C(u_j|u_1,\ldots,u_{j-1})=\left(\frac{\sum_{i=1}^{j}u_i^{-\theta}-j+1}{\sum_{i=1}^{j-1}u_i^{-\theta}-j+2}\right)^{-1/\theta-j+1}=\left(\frac{u_j^{-\theta}+s_{j-1}}{1+s_{j-1}}\right)^{-1/\theta-j+1}$$

where $s_{j-1}=\sum_{i=1}^{j-1}u_i^{-\theta}-(j-1)$. A Monte Carlo simulation of U_j^* given $U_1^*,\ldots,U_{j-1}^*$ is therefore generated by drawing another uniform V^* and solving the equation

$$\left(\frac{(U_j^*)^{-\theta}+S_{j-1}^*}{1+S_{j-1}^*}\right)^{-1/\theta-j+1}=V^*\qquad\text{where}\qquad S_{j-1}^*=\sum_{i=1}^{j-1}(U_i^*)^{-\theta}-(j-1).$$

The expression for U_j^* in Algorithm 6.4 follows from this.

Bibliographical notes

The interest in copulas exploded around the turn of the century, although Mikosch (2006) raises a skeptical eye. General introductions for insurance and finance are Embrechts, P, Lindskog and McNeil (2003) and Cherbini, Lucciano and Vecchiato (2004); see also Nelsen (2006) for a more mathematical treatment or even Joe (1997) for a broader angle on dependence modelling. You will find a good discussion of how Archimedean copulas are simulated in Whelan (2004); see also Frees and Valdez (1998). For applications in insurance consult (among others) Klugman and Parsa (1999), Carriére (2000), Venter (2003), Bäuerle and Grüber (2005) and Escarela and Carriére (2007).

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Exercises

Exercise 6.7.1 Introduce the copulas

$$C^{\min}(u_1,u_2)=\max(u_1+u_2-1,0) \qquad \text{and} \qquad C^{\max}(u_1,u_2)=\min(u_1,u_2).$$

where $0 < u_1, u_2 < 1$ **a)** Argue that $C^{\min}(u_1,u_2)$ is the copula when $U_2 = 1 - U_1$. **b)** Also argue that $C^{\max}(u_1,u_2)$ corresponds to $U_2 = U_1$. **c)** Verify that any copula $C(u_1,u_2)$ satisfies

$$C^{\min}(u_1,u_2) \le C(u_1,u_2) \le C^{\max}(u_1,u_2), \qquad 0 \le u_1,u_2 \le 1.$$

This is known as the **Frechet-Hoeffding** inequality and shows that $C^{\min}(u_1,u_2)$ is a minimum and $C^{\max}(u_1,u_2)$ a *maximum* copula. [**Hint**: For the upper bound note that $C(u_1,u_2) \le \Pr(U_i \le u_i)$ and for the lower one introduce $H(u_1) = C(u_1,u_2) - (u_1+u_2-1)$ for which $H(1) = 0$ and $dH(u_1)/du_1 = \Pr(U_2 \le u_2|u_1) - 1 \le 0$.].

Exercise 6.7.2 Consider the Clayton copula $C(u_1,u_2) = \max(u_1^{-\theta} + u_2^{-\theta}, 0)^{-1/\theta}$ where $\theta \ge -1$. **a)** Argue that it coincides with the minimum copula $C^{\min}(u_1,u_2)$ when $\theta = -1$. **b)** Show that it converges to the maximum copula $C^{\max}(u_1,u_2)$ as $\theta \to \infty$ [**Hint**: The Clayton copula may for positive θ be written $\exp\{\log(u_1^{-\theta} + u_2^{-\theta} - 1)/\theta\}$ and l'Hôpital's rule applied to the exponent.]. The Clayton copula covers in this sense the entire range of dependency.

Exercise 6.7.3 Show that the Clayton copula approaches the independent copula u_1u_2 as $\theta \to 0$ [**Hint**: Use the argument of Exercise 6.7.2b.].

Exercise 6.7.4 **a)** For the Archimedean copula with generator $\phi(u) = (1-u)^3$ verify that $C(u_1, u_2) = 0$ whenever $u_2 \leq \{1-(1-u_1)^3\}^{1/3}$. Consider a general Archimedean copula with monotone decreasing generator $\phi(u)$ with finite $\phi(0)$. **b)** Show that the copula $C(u_1, u_2) = 0$ whenever $\phi(u_1) + \phi(u_2) > \phi(0)$ which is the same as $u_2 > \phi^{-1}\{\phi(0) - \phi(u_1)\}$.

Exercise 6.7.5 Let $R_1 = e^{\xi+\sigma\varepsilon} - 1$ and $R_2 = e^{\xi+\sigma\varepsilon_2} - 1$ where ϵ_1 and ϵ_2 are $N(0,1)$ and Clayton-dependent with parameter θ. **a)** Write a program which which generates m simulations of (R_1, R_2) [**R-commands:** *U1=runif(m); S=U1**(-θ)-1; U2=(1+S)*runif(m)**(θ/(1+θ))-S; R1=exp(ξ+σ*qnorm(U1))-1; R2=exp(ξ + σ*qnorm(U2))-1.*]. **b)** Run the program when $\xi = 0.05$, $\sigma = 0.25$ and $\theta = 0.1$ using $m = 100$ and scatterplot to inspect the pattern [**R-command:** *plot(R1,R2)*. **c)** Repeat b) when $\theta = 2$ and $\theta = 10$ and note how the pattern changes

Exercise 6.7.6 A popular model is the **Franck** copula which reads

$$C(u_1, u_2) = \frac{1}{\theta}\log\left(1 + \frac{(e^{\theta u_1}-1)(e^{\theta u_2}-1)}{e^{\theta}-1}\right), \qquad 0 < u_1, u_2 < 1$$

where θ may be any real number. **a)** Check that the copula conditions are satisfied. **b)** Show that the independent copula emerges as $\theta \to 0$ [**Hint**: Use the approximation $e^x \doteq 1 + x$, valid for small x, to write $C(u_1, u_2) \doteq \log\{1 + (\theta u_1)(\theta_2 u_2)/\theta\}/\theta \to u_1 u_2$ as $\theta \to 0$.].

Exercise 6.7.7 Verify that the Franck copula of the preceding exercise tends to the the maximum copula $C^{\max}(u_1, u_2)$ as $\theta \to -\infty$ and the minimum one $C^{\min}(u_1, u_2)$ as $\theta \to \infty$. [**Hint**: Argue that $e^\theta - 1 \doteq -1$ when $\theta \to -\infty$ so that $C(u_1, u_2) \doteq \log(-e^{-\theta(u_1+u_2)} + e^{\theta u_1} + e^{\theta u_1})/\theta$ which yields $C(u_1, u_2) \doteq u_1 + \log(-e^{\theta u_2} + 1 - e^{\theta(u_2-u_1)}) \to u_1$ when $u_1 < u_2$. When $\theta \to \infty$, justify that $C(u_1, u_2) \doteq \log(1 + e^{\theta(u_1+u_2-1)}) \to \max(u_1 + u_2 - 1, 0)$.].

Exercise 6.7.8 **a)** Differentiate the Franck copula and show that

$$\frac{\partial C(u_1, u_2)}{\partial u_1} = \frac{e^{\theta u_1}(e^{\theta u_2}-1)}{(e^{\theta}-1)(e^{\theta u_1}-1)(e^{\theta u_2}-1)}.$$

b) Use this and the inversion algorithm to justify the Franck sampler

$$U_2^* \leftarrow \frac{1}{\theta}\log\left(1 + \frac{V^*(e^\theta - 1)}{V^* + e^{\theta U_1^*}(1-V^*)}\right) \qquad \text{where} \qquad U_1^*, V^* \sim \text{ uniform.}$$

c) Implement this algorithm so that m simulations are generated [**R-commands:** *U1=runif(m); V=runif(m); Y=V*(e**θ-1)(V+e**(θ*U1)*(1-V)); U2=log(1+Y).*]. **d)** Generate $m = 100$ simulations of (U_1, U_2) when $\theta = -10$ and $\theta = 10$ and scatterplot them [**R-command:** *plot(U1,U2).*]

Exercise 6.7.9 Replace Clayton-dependence with Franck-dependence in the model for (R_1, R_2) in Exercise 6.7.5 **a)** Revise the simulation program accordingly [**R-commands:** Take *U1* and *U2* from Exercise 6.7.8.c) and use *R1=exp(ξ + σ*qnorm(U1))-1; R2=exp(ξ + σ*qnorm(U2))-1.*]. **b)** Use $\theta = -10$ and $\theta = 10$ and scatterplot $m = 100$ Monte Carlo samples when $\xi = 0.05$ and $\sigma = 0.25$ as in Exercise 6.7.5 [**R-command:** *plot(R1,R2).*]. **c)** Redo b) when $\theta = -5$ and $\theta = 5$. **d)** Compare the patterns you have seen with those in Exercise 6.7.5 and speculate which might be most appropriate for financial risk