UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK4530 — SOLUTIONS

Day of examination: 15 December 2017

Examination hours: 14.30 – 18.30

This problem set consists of 4 pages.

Appendices: None

Permitted aids: None.
Godkjent kalkulator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

1a

Ito gives

$$d(e^{bt}r(t)) = be^{bt}r(t)dt + e^{bt}dr(t) = ae^{bt}dt + \sigma e^{bt}d\widetilde{W}(t)$$

Hence,

$$r(t) = r_0e^{-bt} + \frac{a}{b}(1 - e^{-bt}) + \sigma e^{-bt}\int_0^t e^{bs}d\widetilde{W}(s)$$

1b

Using independent increment of $\widetilde{W}$, we find that the characteristic function of $\int_0^t e^{bs}d\widetilde{W}(s)$ is

$$\mathbb{E}_Q[\exp(ix\int_0^t e^{bs}d\widetilde{W})] = \exp(-\frac{1}{2}x^2\int_0^t e^{2bs}ds)$$

Hence, normality of $r(t)$ under Q follows, with mean $r_0e^{-bt} + (a/b)(1 - e^{-bt})$ and variance $(\sigma^2/2b)(1 - e^{-2bt})$.

1c

Integrating from t to T gives,

$$e^{bT}r(T) - e^{bt}r(t) = a\int_t^T e^{bs}ds + \sigma\int_t^T e^{bs}d\widetilde{W}(s)$$

Hence,

$$r(T) = e^{-b(T-t)}r(t) + \frac{a}{b}(1 - e^{-b(T-t)}) + \sigma e^{-bT}\int_t^T e^{-bs}d\widetilde{W}(s)$$

(Continued on page 2.)

From $dr(t)$, we also find

$$r(T) - r(t) = a(T-t) - b\int_t^T r(s)ds + \sigma(\widetilde{W}(T) - \widetilde{W}(t))$$

Re-arraging, and inserting $r(T)$,

$$\int_t^T r(s)ds = \frac{1}{b}(1-e^{-b(T-t)})r(t) - \frac{a}{b^2}(1-e^{-b(T-t)}) - \frac{\sigma}{b}e^{-bT}\int_t^T e^{bs}d\widetilde{W}(s) + \frac{a}{b}(T-t) + \frac{\sigma}{b}\int_t^T d\widetilde{W}(s)$$

$r(t)$ is $\mathcal{F}_t$-measurable, and appealing to independent increment property of $\widetilde{W}$, we find

$$\mathbb{E}_Q[e^{-\int_t^T r(s)\,ds}\,|\,\mathcal{F}_t] = \exp(A(t,T) - C(t,T)r(t))$$

where

$$A(t,T) = -\frac{a}{b}(T-t) + \frac{a}{b^2}(1-e^{-b(T-t)}) + \frac{\sigma^2}{2b}\int_t^T (1-e^{-2b(T-s)})^2 ds$$

and

$$C(t,T) = \frac{1}{b}(1-e^{-b(T-t)})$$

for $T \geq t$

1d

$P(t,T)/B(t)$ a Q-martingale, yields, since $P(T,T) = 1$,

$$\mathbb{E}_Q[1/B(T)|\mathcal{F}_t] = P(t,T)/B(t)$$

Hence, claim holds since $B(t)$ is $\mathcal{F}_t$-measurable. The conditional expectation is well-defined as r and its time integral are normal, and hence has finite exponential moments.

1e

We have

$$d(P(t,T)/B(t)) = d(e^{-\int_0^t r(s)ds}P(t,T))$$

We know this is a Q-martingale, so the dynamics will only involve the $d\widetilde{W}$-term. This we get from Ito's formula from the $d\widetilde{W}(t)$-term in $dP(t,T)$, which from above, is

$$-P(t,T)C(t,T)\sigma d\widetilde{W}(t)$$

Hence,

$$d(P(t,T)/B(t)) = -\sigma C(t,T)(P(t,T)/B(t))d\widetilde{W}(t)$$

P-dynamics: Let γ be an Ito integrable process such that the stochastic exponential of $\int_0^t \gamma(s)d\widetilde{W}(s)$ is a Q-martingale. Then by Girsanov there exists a P with dP/dQ and a W being a P-Brownian motion,

$$dW(t) = d\widetilde{W}(t) - \gamma(t)dt$$

Inserting for dW in the dynamics of $P(t,T)/B(t)$ yields the P-dynamics.

(Continued on page 3.)

Problem 2

2a

For a volatility $\bar{\sigma}(t,T)$, the no-arbitrage dynamics of the forward is given by

$$f(t,T) = f(0,T) + \int_0^t \bar{\sigma}(s,T) \int_s^T \bar{\sigma}(s,u)^{\mathrm{Tr}} du ds + \int_0^t \bar{\sigma}(s,T) d\widetilde{W}(s)$$

Inserting the defined volatility in this exercise, yields,

$$f(t,T) = f(0,T) + \int_0^t (T-s)\|\sigma(s)\|^2 ds + \int_0^t \bar{\sigma}(s,T) d\widetilde{W}(s)$$

where $\|\cdot\|$ is the 2-norm in $\mathbb{R}^n$.

2b

We have $r(t) = f(t,t)$, thus

$$r(t) = f(0,t) + \int_0^t (t-s)\|\sigma(s)\|^2 ds + \int_0^t \sigma(s) d\widetilde{W}(s)$$

The students can also find $dr(t)$.

We know that

$$P(t,T) = \exp(-\int_t^T f(t,u)du)$$

By (stochastic) Fubini

$$\int_t^T f(t,u)du = \int_t^T f(0,u)du + \frac{1}{2}\int_0^t ((T-s)^2 - (t-s)^2)\|\sigma(s)\|^2 ds + (T-t)\int_0^t \sigma(s) d\widetilde{W}(s)$$

Students can use Ito's formula to derive the dynamics of $dP(t,T)$ from this expression.

2c

By Bayes' Theorem, it holds,

$$\mathbb{E}_{Q^T}[X|\mathcal{F}_t] = \frac{\mathbb{E}_Q\left[X\frac{dQ^T}{dQ}\right]}{\mathbb{E}_Q\left[\frac{dQ^T}{dQ}\right]} = \frac{\mathbb{E}_Q\left[X\frac{1}{P(0,T)B(T)}|\mathcal{F}_t\right]}{\frac{P(t,T)}{P(0,T)B(t)}} = \frac{B(t)}{P(t,T)}\mathbb{E}_Q\left[XB^{-1}(T)|\mathcal{F}_t\right]$$

using Q-martingale property of $P(t,T)/B(t)$. The result follows after using that $B(t)$ is $\mathcal{F}_t$-measurable.

Problem 3

3a

THis is GBM, with Q^{T_2}-dynamics,

$$L(t,T_1) = L(0,T_1)\exp\left(\lambda W^{T_2}(t) - \frac{1}{2}\lambda^2 t\right)$$

for $t \le T_1$. W^{T_2} is Q^{T_2}-BM, hence $L(t,T_1)$ is lognormal under Q^{T_2} and thus also Q^{T_2}-integrable. Using independent increment propery of Brownian motion shows that $t \mapsto L(t,T_1)$ is a Q^{T_2}-martingale.

(Continued on page 4.)

3b

Consider

$$\mathcal{E}_t(\sigma_{T_1,T_2} \bullet W^{T_2}) = \exp\left(\int_0^t \sigma_{T_1,T_2}(s)dW^{T_2}(s) - \frac{1}{2}\int_0^t \sigma^2_{T_1,T_2}(s)ds\right)$$

Since L is positive,

$$|\sigma_{T_1,T_2}(t)| \leq \lambda$$

and Novikov's condition ensure that $t \mapsto \mathcal{E}_t(...)$ is a true Q^{T_2}-martingale. We have

$$\frac{dQ^{T_1}}{dQ^{T_2}}|_{\mathcal{F}_t} = \mathcal{E}_t\left(\sigma_{T_1,T_2} \bullet W^{T_2}\right)$$

and by Girsanov's theorem we find that Q^{T_1} is a probability measure where W^{T_1} is a Brownian motion.

We find

$$dL(t,T_1) = \lambda L(t,T_1)(dW^{T_1}(t) + \sigma_{T_1,T_2}(t)dt)$$

and since σ_{T_1,T_2} is stochastic, $L(t,T_1)$ is not lognormal wrt Q^{T_1}.

3c

Under Q^{T_2}, we have

$$L(T_1,T_1) = L(t,T_1)\exp\left(\lambda(W^{T_2}(T_1) - W^{T_2}(t)) - \frac{1}{2}\lambda^2(T_1 - t)\right)$$

We find,

$$E_{Q^{T_2}}\left[\max(L(T_1,T_1) - \kappa, 0)\,|\,\mathcal{F}_t\right] = \mathbb{E}_{Q^{T_2}}\left[L(T_1,T_1)1(L(T_1,T_1) > \kappa)|\mathcal{F}_t\right] - \kappa Q^{T_2}(L(T_1,T_1) > \kappa|\mathcal{F}_t)$$

Using that $L(T_1,T_1)$ is conditionally lognormal, we can derive as in Black & Scholes the formula for the price, being

$$\pi(t) = P(t,T_2)\left(L(t,T_1)\Phi(d_1) - \kappa\Phi(d_2)\right)$$

where '*phi* is cumulative normal distribution function and

$$d_{1,2} = \frac{\ln(L(t,T_1)/\kappa) \pm \frac{1}{2}\lambda^2(T_1 - t)}{\lambda\sqrt{T_1 - t}}$$

END