Non-life insurance mathematics

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Overview

	Important issues	Models treated	Curriculum	Duration (in lectures)
done	What is driving the result of a non-life insurance company?	insurance economics models	Lecture notes	0,5
done	How is claim frequency modelled?	Poisson, Compound Poisson and Poisson regression	Section 8.2-4 EB	1,5
done	How can claims reserving be modelled?	Chain ladder, Bernhuetter Ferguson, Cape Cod,	Note by Patrick Dahl	2
done	How can claim size be modelled?	Gamma distribution, log-normal distribution	Chapter 9 EB	2
done	How are insurance policies priced?	Generalized Linear models, estimation, testing and modelling. CRM models.	Chapter 10 EB	2
	Credibility theory	Buhlmann Straub	Chapter 10 EB	1
	Reinsurance		Chapter 10 EB	1
done	Solvency		Chapter 10 EB	1
	Repetition			1

Credibility theory was used before computers arrived

Motivation

Linear credibility

Example

Optimal credibility

The need for credibility theory

- So far we have studied how GLMs may be used to estimate relativities for rating factors with a moderate number of levels.
  - The rating factors were either categorical with few levels (e.g., gender), or
  - formed by grouping of a continuous variable (e.g., policy holder age)
- In case of insufficient data levels can be merged
- However, for categorical rating factors with a large number of levels without an inherent ordering there is no simple way to form groups to solve the problem with insufficient data

Example 1: car insurance

- Car model is an important rating factor in motor insurance
  - Exlusive cars are more attractive to thieves and more expensive to repair than common cars
  - Sports cars may be driven differently (more recklessly?) than family cars
- In Norway there are more than 3000 car model codes
- How should these be grouped?

Example 2: Experience rating

- The customer is used as a rating factor
- The *credibility estimators* are used to calibrate the pure premium
- The credibility estimators are weighted averages of
  - the pure premium based on the individual claims experience
  - The pure premium for the entire portfolio of insurance policies

Example 3: Geographic zones

- How should the premium be affected by the area of residence? (parallel to the car model example)
- Example: postcode. There are over 8000 postcodes in Norway
- We can merge neighbouring areas
- But they may have different risk characteristics

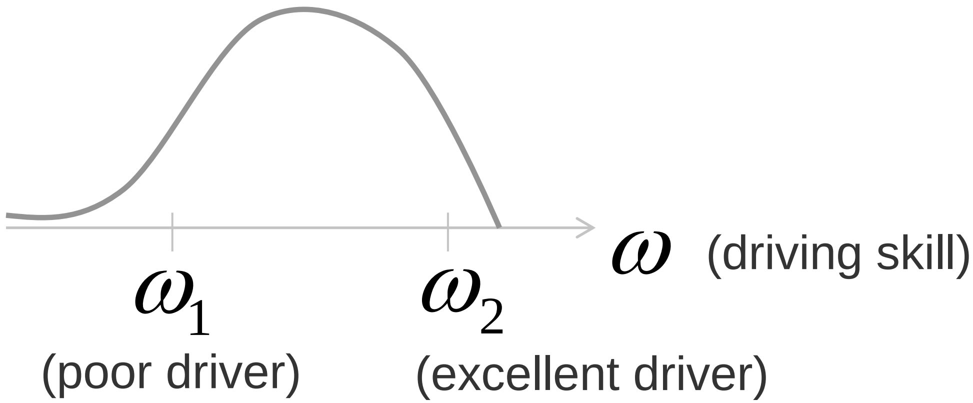
The credibility approach

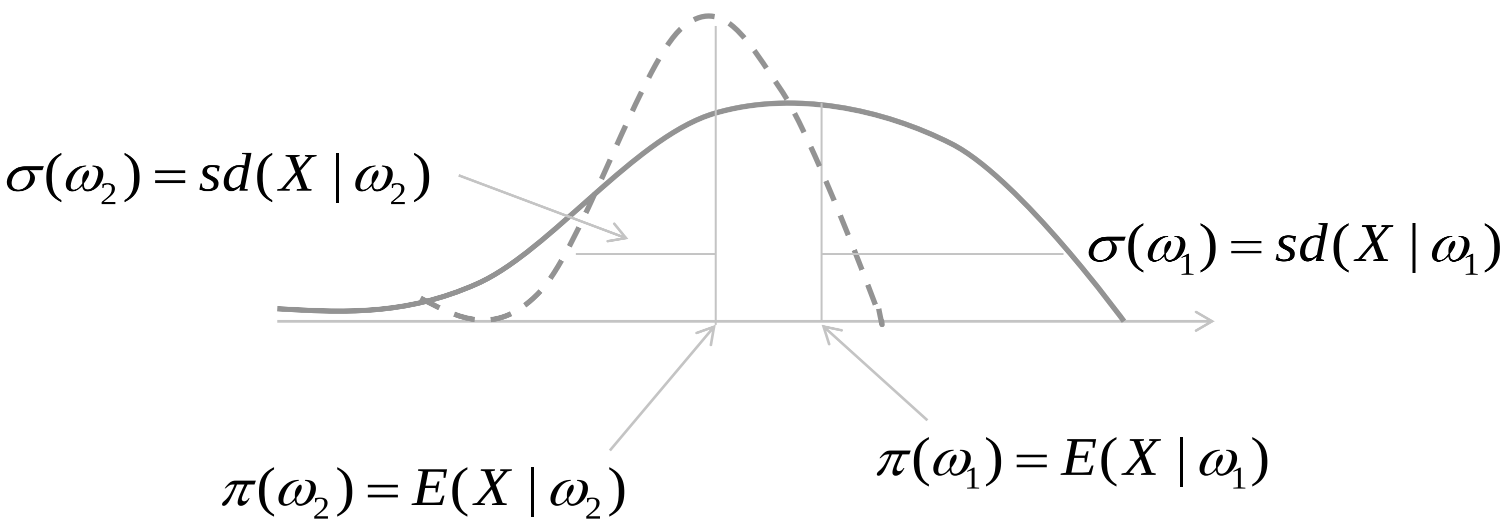
- The basic assumption is that policy holders carry a list of attributes with impact on risk
- The parameter ω could be how the car is used by the customer (degree of recklessness) or for example driving skill
- It is assumed that ω exists and has been drawn randomly for each individual
- X is the sum of claims during a certain period of time (say a year) and introduce

$$\pi(\omega) = E(X \mid \omega) \quad \text{and} \quad \sigma(\omega) = \text{sd}(X \mid \omega)$$

- We seek $\pi = \pi(\omega)$ the conditional pure premium of the policy holder as basis for pricing
- On group level there is a common ω that applies to all risks jointly
- We will focus on the individual level here, as the difference between individual and group is minor from a mathematical point of view

Omega is random and has been drawn for each policy holder





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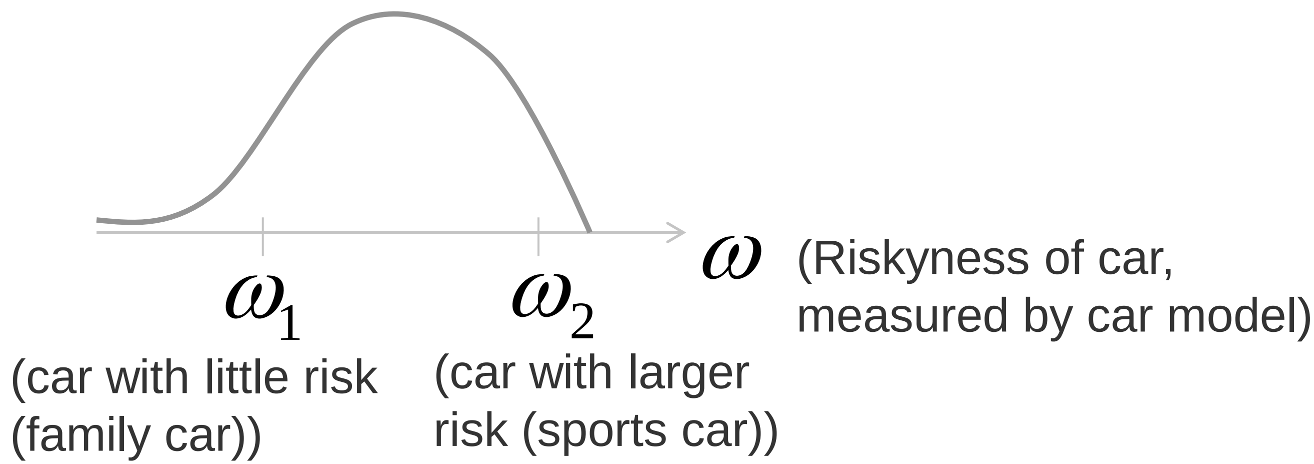

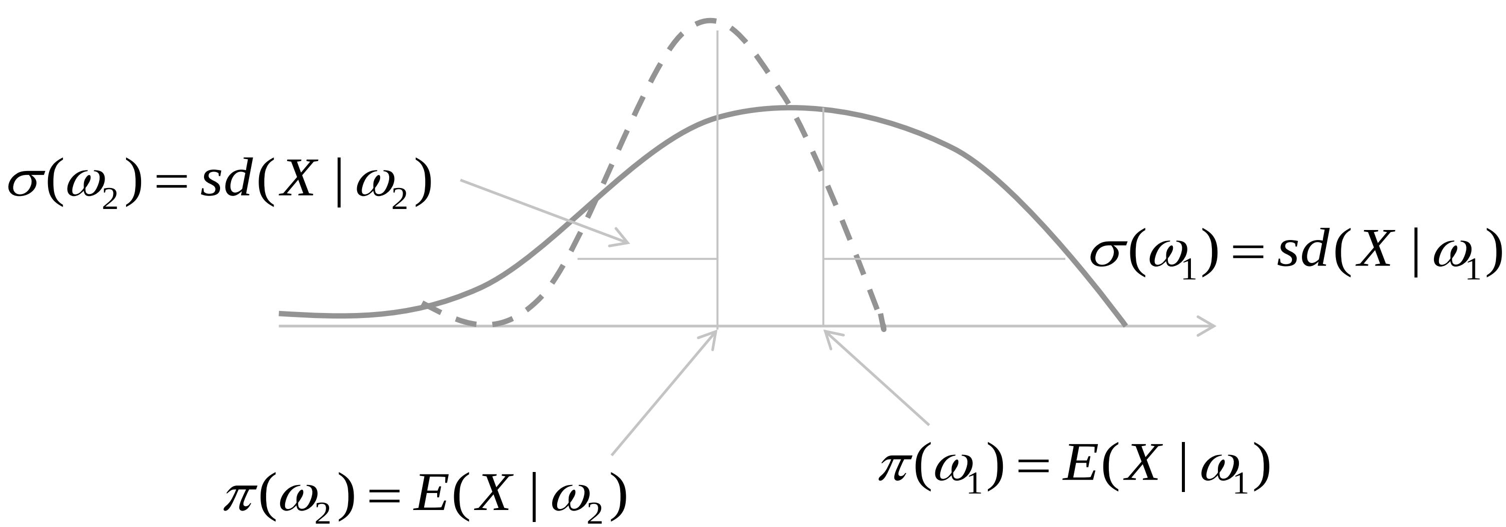

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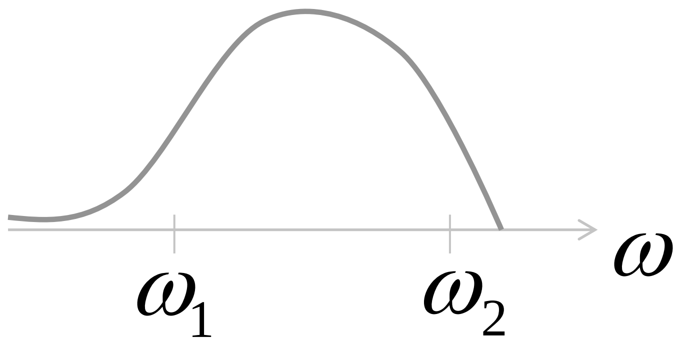
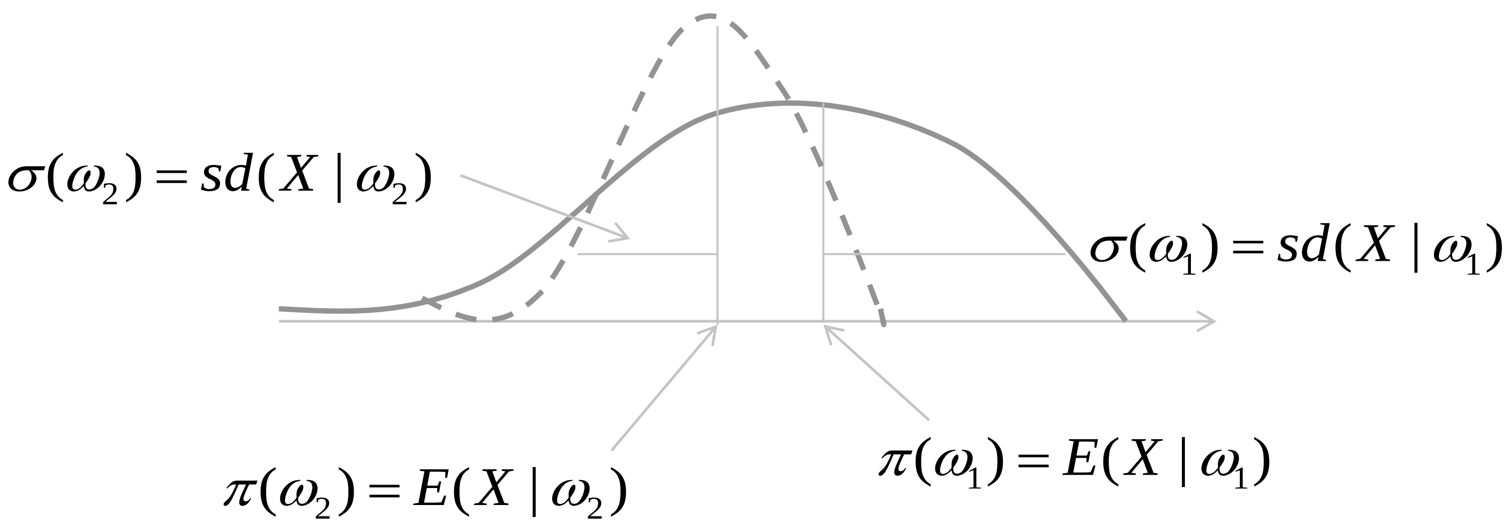


(area with good climate)

(area with poor climate)

(Riskyness of geographical area, measured by post code)



Goal

We want to estimate

$$\bar{\pi}(\omega) = E\{E(X \mid \omega)\},\ \mathrm{E}\{\sigma^2(\omega)\} = E\{sd^2(X \mid \omega)\} \text{ and } \operatorname{var}\pi(\omega)$$

These are called structural parameters.
If $X_1,\ldots,X_K$ are realizations from X, where X is the sum of claims during a year we want the following expression

$$\hat{\pi}_K = (1-w)\bar{\pi} + w\bar{X}_K$$

- We want to find what w is in two situations.
- The weight w defines a compromise between the average pure premium $\bar{\pi}$ of the population and the track record of the policy holder.

Linear credibility

- The standard method in credibility is the linear one with estimates of pi of the form

$$\hat{\pi}_K = b_0 + b_1 X_1 + ... + b_K X_K$$

where $b_0,\dots,b_K$ are coefficients tailored to minimized the mean squared error $E(\hat{\pi}_K - \pi)^2$

- The fact that $X, X_1,\dots,X_K$ are conditionally independent with the same distribution forces $b_1=\dots=b_K$, and if w/K is their common value, the estimate becomes

$$\hat{\pi}_K = b_0 + w\bar{X}_K \quad \text{where} \quad \bar{X}_K = (X_1 + ... + X_K)/K$$

Linear credibility

- To proceed we need the so-called *structural parameters*

$$\bar{\pi} = E\{\pi(\omega)\}, \quad \nu^2 = \text{var}\{\pi(\omega)\}, \quad \tau^2 = E\{\sigma^2(\omega)\}$$

where $\bar{\pi}$ is the average pure premium for the entire population.

- It is also the expectation for individuals since by the rule of double expectation

$$E(X) = E\{E\{X \mid \omega\}\} = E\{\pi(\omega)\} = \bar{\pi}$$

- Both ν and τ represent variation. The former is caused by diversity between individuals and the latter by the physical processes behind the incidents.Their impact on var(X) can be understood through the rule of double variance, i.e.,

$$\text{var}(X) = E\{\text{var}(X \mid \omega)\} + \text{var}\{E(X \mid \omega)\}$$

$$= E\{\sigma^2(\omega)\} + \text{var}\{\pi(\omega)\} = \tau^2 + \nu^2$$

and ν^2 and τ^2 represent uncertainties of different origin that add to var(X)

Linear credibility

- The optimal linear credibility estimate now becomes

$$\hat{\pi}_K = (1-w)\bar{\pi} + w\bar{X}_K, \quad \text{where} \quad w = \frac{\nu^2}{\nu^2 + \tau^2 / K}$$

which is proved in Section 10.7, where it is also established that

$$E(\hat{\pi}_K - \pi) = 0 \quad \text{and that} \quad \text{sd}(\hat{\pi}_K - \pi) = \frac{\nu}{\sqrt{1 + K\nu^2/\tau^2}}.$$

- The estimate is ubiased and its standard deviation decreases with K
- The weight w defines a compromise between the average pure premium pi bar of the population and the track record of the policy holder
- Note that w=0 if K=0; i.e. without historical information the best estimate is the population average

Estimation of structural parameters

- Credibility estimation is based on parameters that must be determined from historical data.
- Historical data for J policies that been in the company for K1,…,KJ years are then of the form

Policies	Annual claims			mean	sd
1	x_{11}	...	x_{1K_1}	$\bar{x}_1$	s_1
.	.		.	.	.
.	.		.	.	.
J	x_{J1}	...	x_{JK_1}	$\bar{x}_J$	s_J

where the j'th row $x_{j1},\ldots,x_{jK_j}$ are the annual claims from client j and $\bar{x}_j$ and s_j their mean and standard deviation

- The following estimates are essentially due to Sundt (1983) and Bühlmann and Straub (1970):

Estimation of structural parameters

Let $\mathrm{K} = K_1 + \ldots + K_J$. Unbiased, moment estimates of the structural parameters are then

$$\hat{\hat{\pi}} = \frac{1}{\mathrm{K}} \sum_{j=1}^{J} K_j \bar{x}_j, \quad \hat{\tau}^2 = \frac{1}{\mathrm{K} - J} \sum_{j=1}^{J} (K_j - 1) s_j^2$$

and

$$\hat{\nu}^2 = \frac{\sum_{j=1}^{J} (K_j / \mathrm{K})(\bar{x}_j - \hat{\hat{\pi}})^2 - \hat{\tau}^2 (J-1)/\mathrm{K}}{1 - \sum_{j=1}^{J} (K_j / \mathrm{K})^2}$$

- For verification see Section 10.7.
- The expression for $\hat{\nu}^2$ may be negative. If it is let $\hat{\nu} = 0$ and assume that the underlying variation is too small to be detected.

The most accurate estimate

- Let $X_1,\ldots,X_K$ (policy level) be realizations of X dating K years back
- The most accurate estimate of pi from such records is (section 6.4) the conditional mean

$$\hat{\pi}_K = E(X \mid x_1,\ldots,x_K)$$

where $x_1,\ldots,x_K$ are the actual values.

- A natural framework is the common factor model of Section 6.3 where $X,X_1,\ldots,X_K$ are identically and independently distributed given omega
- This won't be true when underlying conditions change systematically
- A problem with the estimate above is that it requires a joint model for $X,X_1,\ldots,X_K$ and omega.
- A more natural framework is to break X down on claim number N and losses per incident Z.
- First linear credibility is considered

Optimal credibility

- The preceding estimates are the best *linear* methods but the Bayesian estimate $\hat{\pi}_K = E(X \mid x_1,...,x_K)$ is optimal among *all* methods and offers an improvement.
- Break the historical record $x_1,\ldots,x_K$ down on annual claim numbers $n_1,\ldots n_K$ and losses $z_1,\ldots,z_n$ where $n=n_1+\ldots+n_K$ and assume independence.
- The Bayes estimate of $\pi = E(X) = E(N)E(Z)$ now becomes

$$\hat{\pi}_K = E(X \mid n_1,...,n_K, z_1,...,z_n) = E(N \mid n_1,...,n_K)E(Z \mid z_1,...,z_n)$$

and there are two parts.

- Often claim intensity muh fluctuates more strongly from one policy holder to another than does the expected loss $\xi_z = E(Z)$.
- We shall disregard all such variation in ξ_z .
- Then $\xi_z = E(Z \mid z_1,...,z_n)$ is independently of $z_1,\ldots,z_n$ and the Bayesian estimate becomes

$$\hat{\pi}_K = \xi_z E(N \mid n_1,...,n_K).$$

Optimal credibility

- A model for past and future claim numbers is needed. The natural one is the common factor model where $N, N_1, \ldots, N_k$ are conditionally independent and identically distributed given muh with N and all N_k being Poisson (mu T).
- For mu assume the standard representation

$$\mu = \xi_\mu G \quad \text{and} \quad G \sim Gamma(\alpha)$$

where E(G)=1. It is shown in Section 10.7 that the estimate $\hat{\pi}_K = \xi_z E(N \mid n_1, \ldots, n_K)$ now becomes

$$\hat{\pi}_K = \bar{\pi} \frac{\bar{n} + \alpha / K}{\xi_\mu T + \alpha / K} \quad \text{where} \quad \bar{\pi} = \xi_\mu T \xi_z$$

$$\text{and } \bar{n} = (n_1 + \ldots + n_K) / K$$

- Note that the population average pi bar is adjusted up or down according to whether the average claim number n bar is larger or smaller than its expectation $\xi_\mu T$. The error (proved in Section 10.7) is

$$E(\hat{\pi}_K - \pi) = 0 \quad \text{and} \quad \text{sd}(\hat{\pi}_K - \pi) = \frac{\bar{\pi}}{\sqrt{\alpha + K \xi_\mu T}}$$