Non-life insurance mathematics

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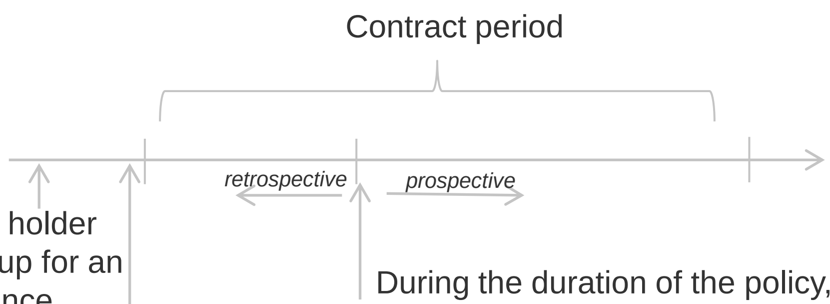
Overview

Non-life insurance from a financial perspective:

for a premium an insurance company commits itself to pay a sum if an event has occured



Policy holder signs up for an insurance

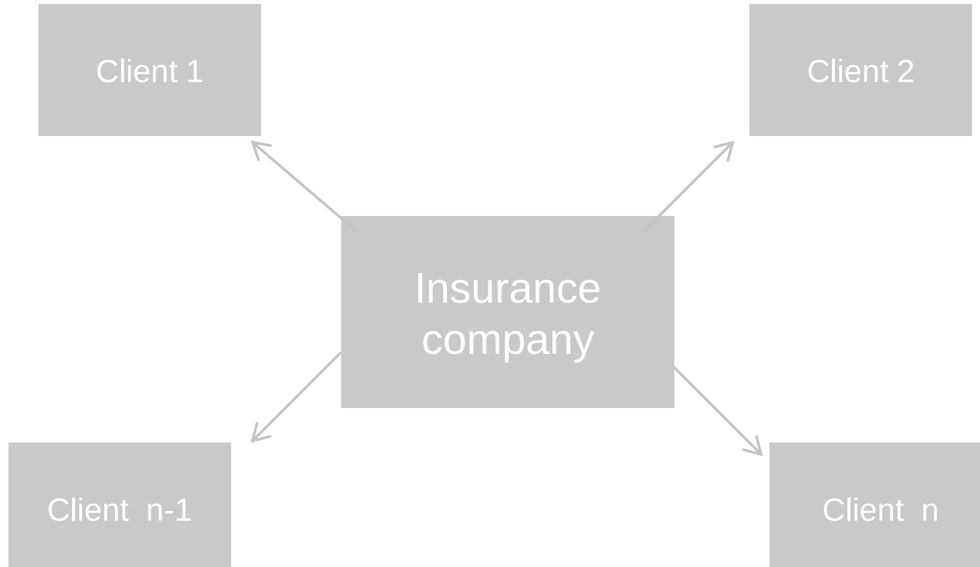
Policy holder pays premium. Insurance company starts to earn premium

During the duration of the policy, some of the premium is earned, some is unearned

- How much premium is earned?
- How much premium is unearned?
- Is the unearned premium sufficient?

Premium reserve, *prospective*

Why does it work??



- Economic risk is transferred from the policyholder to the insurer
- Due to the law of large numbers (many almost independent clients), the loss of the insurance company is much more predictable than that of an individual
- Therefore the premium should be based on the expected loss that is transferred from the policyholder to the insurer

Much of the course is about computing this expected loss ...but first some insurance economics

Insurance mathematics is fundamental in insurance economics

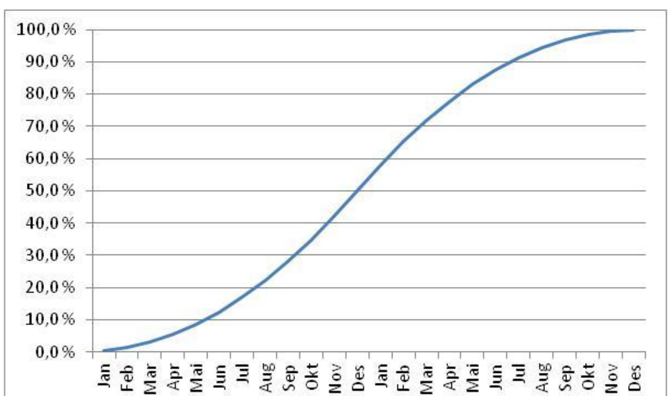
The result drivers of insurance economics:

Result elements:	**Result drivers:**
+ Insurance premium	Risk based pricing, reinsurance
+ financial income	International economy for example interest rate level, risk profile for example stocks/no stocks
- claims	risk reducing measures (for example installing burglar alarm), risk selection (client behaviour), change in legislation, weather phenomenons, demographic factors, reinsurance
- operational costs	measures to increase operational efficiency, IT-systems, wage development
= result to be distributed among the owners and the	Tax politics

Premium income

- Earning of premium adjustments take 2 years in non-life insurance:

	Maturity pattern	Year 1 Jan	Year 1 Feb	Year 1 Mar	Year 1 Apr	Year 1 May	Year 1 Jun	Year 1 Jul	Year 1 Aug	Year 1 Sep	Year 1 Oct	Year 1 Nov	Year 1 Dec	Year 2 Jan	Year 2 Feb	Year 2 Mar	Year 2 Apr	Year 2 May	Year 2 Jun	Year 2 Jul	Year 2 Aug	Year 2 Sep	Year 2 Oct	Year 2 Nov	Year 2 Dec
January	8 %	0,3 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,3 %											
February	8 %		0,3 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,3 %										
March	8 %			0,3 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,3 %									
April	8 %				0,3 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,3 %								
May	8 %					0,3 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,3 %							
June	8 %						0,3 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,3 %						
July	8 %							0,3 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,3 %					
August	8 %								0,3 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,3 %				
September	8 %									0,3 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,3 %			
October	8 %										0,3 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,3 %		
November	8 %											0,3 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,3 %	
December	8 %												0,3 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,7 %	0,3 %
Sum		0 %	1 %	3 %	6 %	9 %	13 %	17 %	22 %	28 %	35 %	42 %	50 %	58 %	65 %	72 %	78 %	83 %	88 %	91 %	94 %	97 %	99 %	100 %	100 %



- Assumes that premium adjustment is implemented January 1st.
- Assumes that the portfolio's maturity pattern is evenly distributed during the year

Loss ratio

- Shows how much of the premium income is spent to cover losses

Amounts in 1 000 000 NOK	2012
Written gross premium	1 450
- ceded reinsurance premium	-270
Change in reserve for unearned gross premium	-110
-change in reinsurance share of unearned premium	25
Net premium income	**1 095**

Amounts in 1 000 000 NOK	2012
Paid claims gross	-870
- Reinsurance share of paid claims gross	120
change in gross claims reserve	-200
-change in reinsurance part of gross claims reserve	100
Net claims costs	**-850**

	Gross	Net
Incurred losses	1070 (-870-200)	850
Earned premium	1340 (1450-110)	1095
Loss ratio	79.9%	77.6%

- What does the difference in loss ratio gross and net tell us?

Overview

Important issues	Models treated	Curriculum	Duration (in lectures)
What is driving the result of a non-life insurance company?	insurance economics models	Lecture notes	0,5
How is claim frequency modelled?	Poisson, Compound Poisson and Poisson regression	Section 8.2-4 EB	1,5
How can claims reserving be modelled?	Chain ladder, Bernhuetter Ferguson, Cape Cod,	Note by Patrick Dahl	2
How can claim size be modelled?	Gamma distribution, log-normal distribution	Chapter 9 EB	2
How are insurance policies priced?	Generalized Linear models, estimation, testing and modelling. CRM models.	Chapter 10 EB	2
Credibility theory	Buhlmann Straub	Chapter 10 EB	1
Reinsurance		Chapter 10 EB	1
Solvency		Chapter 10 EB	1
Repetition			1

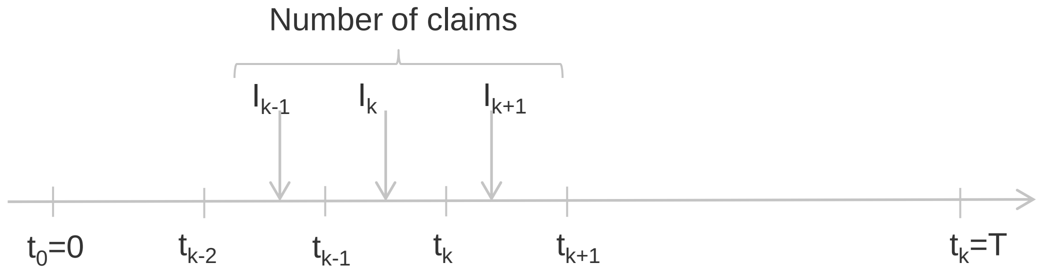
Overview of this session

- The Poisson model (Section 8.2 EB)
- Some important notions and some practice too
- Examples of claim frequencies
- Random intensities (Section 8.3 EB)

Introduction

- Pure premium = likelihood of claim event (claims frequency) * economic consequence of claim event (claim severity)
- What is the likelihood of a claim event?
- It depends!!....on
  - risk exposure (extent and nature of use)
  - object characteristics (quality and nature of object)
  - subject characteristics (behaviour of user)
  - geographical characteristics (for example weather conditions and traffic complexity)
- These dependencies are normally handled through regression, where the number of claims is the response and the factors above are the explanatory variables
- Let us start by looking at the Poisson model

The world of Poisson (Chapter 8.2)



- What is rare can be described mathematically by cutting a given time period T into K small pieces of equal length h=T/K
- On short intervals the chance of more than one incident is remote
- Assuming no more than 1 event per interval the count for the entire period is

$$N=I_1+\dots+I_K, \qquad \text{where } I_j \text{ is either 0 or 1 for } j=1,\dots,K$$

- If $p=\Pr(I_k=1)$ is equal for all k and events are independent, this is an ordinary Bernoulli series

$$\Pr(N=n)=\frac{K!}{n!(K-n)!}p^n(1-p)^{K-n}, \quad \text{for} \quad n=0,1,\dots,K$$

- Assume that p is proportional to h and set $p=\mu h$ where μ is an intensity which applies per time unit

The world of Poisson

$$\Pr(N=n)=\frac{K!}{n!(K-n)!}\,p^n(1-p)^{K-n}$$

$$=\frac{K!}{n!(K-n)!}\left(\frac{\mu T}{K}\right)^n\left(1-\frac{\mu T}{K}\right)^{K-n}$$

$$=\frac{(\mu T)^n}{n!}\;\underbrace{\frac{K(K-1)\cdots(K-n+1)}{K^n}}_{\xrightarrow[K\to\infty]{}1}\;\underbrace{\left(1-\frac{\mu T}{K}\right)^K}_{\xrightarrow[K\to\infty]{}e^{-\mu T}}\;\underbrace{\frac{1}{\left(1-\frac{\mu T}{K}\right)^n}}_{\xrightarrow[K\to\infty]{}1}$$

$$\Rightarrow \Pr(N=n)\xrightarrow[K\to\infty]{}\frac{(\mu T)^n}{n!}e^{-\mu T}$$

In the limit N is Poisson distributed with parameter $\lambda=\mu T$

The world of Poisson

- Let us proceed removing the zero/one restriction on I_k. A more flexible specification is

$$\Pr(I_k = 0) = 1 - \mu h + o(h), \quad \Pr(I_k = 1) = \mu h + o(h), \quad \Pr(\mathrm{I}_k > 1) = o(h)$$

Where o(h) signifies a mathematical expression for which

$$\frac{o(h)}{h} \to 0 \quad \text{as} \quad h \to 0$$

It is verified in Section 8.6 that o(h) does not count in the limit

Consider a portfolio with J policies. There are now J independent processes in parallel and if μ_j is the intensity of policy j and $\mathbf{I}_k$ the total number of claims in period k, then

$$\Pr(\mathbf{I}_k = 0) = \underbrace{\prod_{j=1}^{J}(1-\mu_j)}_{\text{No claims}} \quad \text{and} \quad \Pr(\mathbf{I}_k = 1) = \sum_{i=1}^{J}\underbrace{\left\{\mu_i h \prod_{j \neq i}(1-\mu_j h)\right\}}_{\text{Claims policy i only}}$$

The world of Poisson

- Both quanities simplify when the products are calculated and the powers of h identified

$$
\begin{aligned}
\Pr(\mathbf{I}_k = 0) &\overset{J=3}{=} \prod_{j=1}^{3}(1-\mu_j h) = (1-\mu_1 h)(1-\mu_2 h)(1-\mu_3 h) \\
&= (1-\mu_1 h - \mu_2 h + \mu_2\mu_1 h^2)(1-\mu_3 h) \\
&= 1-\mu_1 h - \mu_2 h + \mu_2\mu_1 h^2 - \mu_3 h(1-\mu_1 h - \mu_2 h + \mu_2\mu_1 h^2) \\
&= 1-\mu_1 h - \mu_2 h - \mu_3 h + o(h)
\end{aligned}
$$

$$\Pr(\mathbf{I}_k = 1) = (\sum_{j=1}^{J}\mu_j)h + o(h)$$

- It follows that the portfolio number of claims **N** is Poisson distributed with parameter

$$\lambda = (\mu_1 + \ldots + \mu_J)T = J\bar{\mu}T, \quad \text{where } \bar{\mu} = (\mu_1 + \ldots + \mu_J)/J$$

- When claim intensities vary over the portfolio, only their average counts

When the intensity varies over time

- A time varying function $\mu = \mu(t)$ handles the mathematics. The binary variables $I_1,...I_k$ are now based on different intensities

$$\mu_1,....,\mu_K \quad \text{where} \quad \mu_\mathrm{k} = \mu(t_k) \quad \text{for} \quad k = 1,...,K$$

- When $I_1,...I_k$ are added to the total count N, this is the same issue as if K different policies apply on an interval of length h. In other words, N must still be Poisson, now with parameter

$$\lambda = h\sum_{k=1}^{K}\mu_k \to \int_0^T \mu(t)dt \quad \text{as} \quad h \to 0$$

where the limit is how integrals are defined. The Poisson parameter for N can also be written

$$\lambda = T\overline{\mu} \quad \text{where} \quad \overline{\mu} = \frac{1}{T}\int_0^T \mu(t)dt,$$

And the introduction of a time-varying function $\mu(t)$ doesn't change things much. A time average $\overline{\mu}$ takes over from a constant μ

The Poisson distribution

- Claim numbers, N for policies and **N** for portfolios, are Poisson distributed with parameters

$$\lambda = \mu T \quad \text{and} \quad \lambda = J\mu T$$

Policy level   Portfolio level

The intensity μ is an average over time and policies.

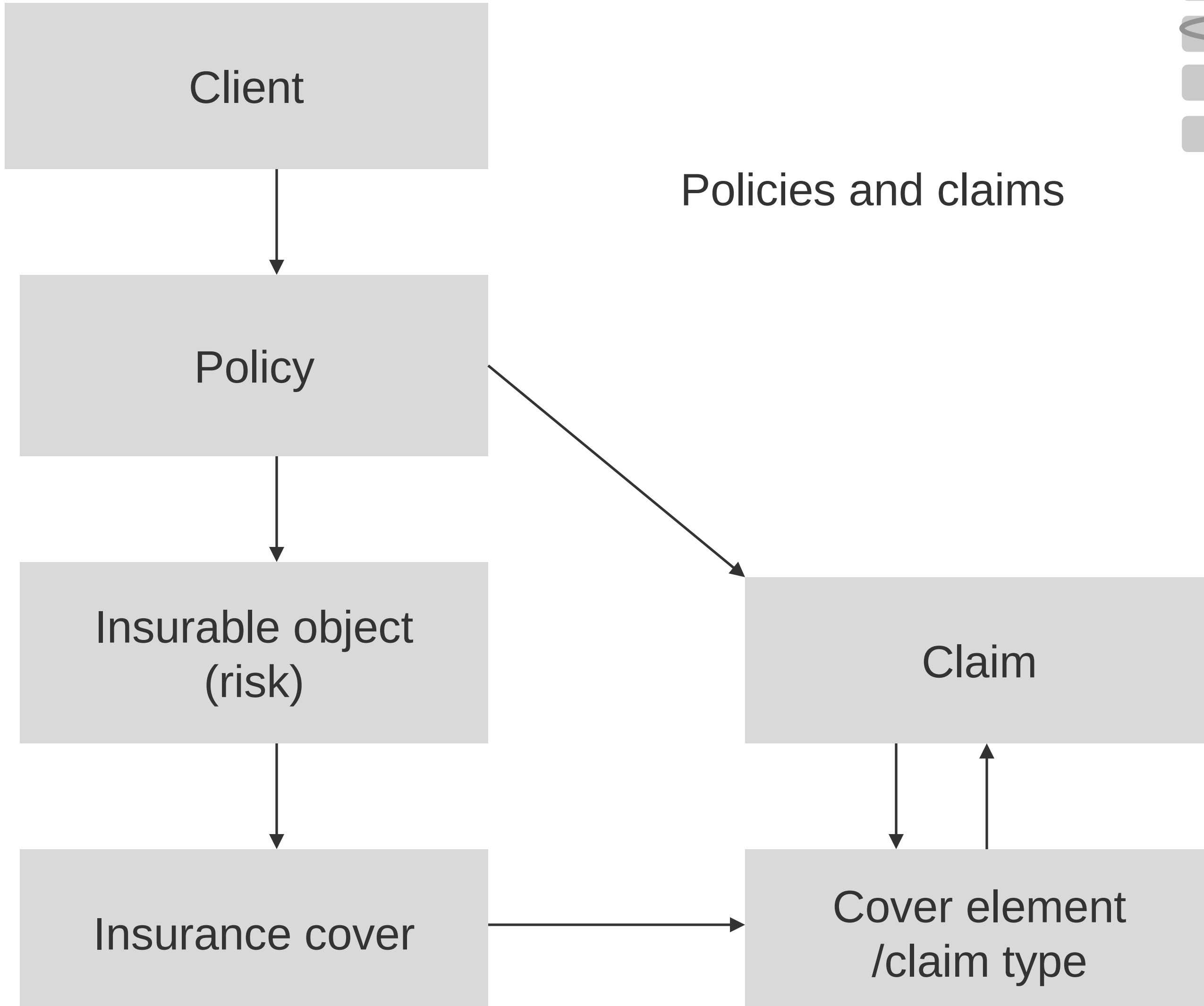
Poisson models have useful operational properties. Mean, standard deviation and skewness are

$$E(N) = \lambda, \quad sd(N) = \sqrt{\lambda} \quad \text{and} \quad skew(\lambda) = \frac{1}{\sqrt{\lambda}}$$

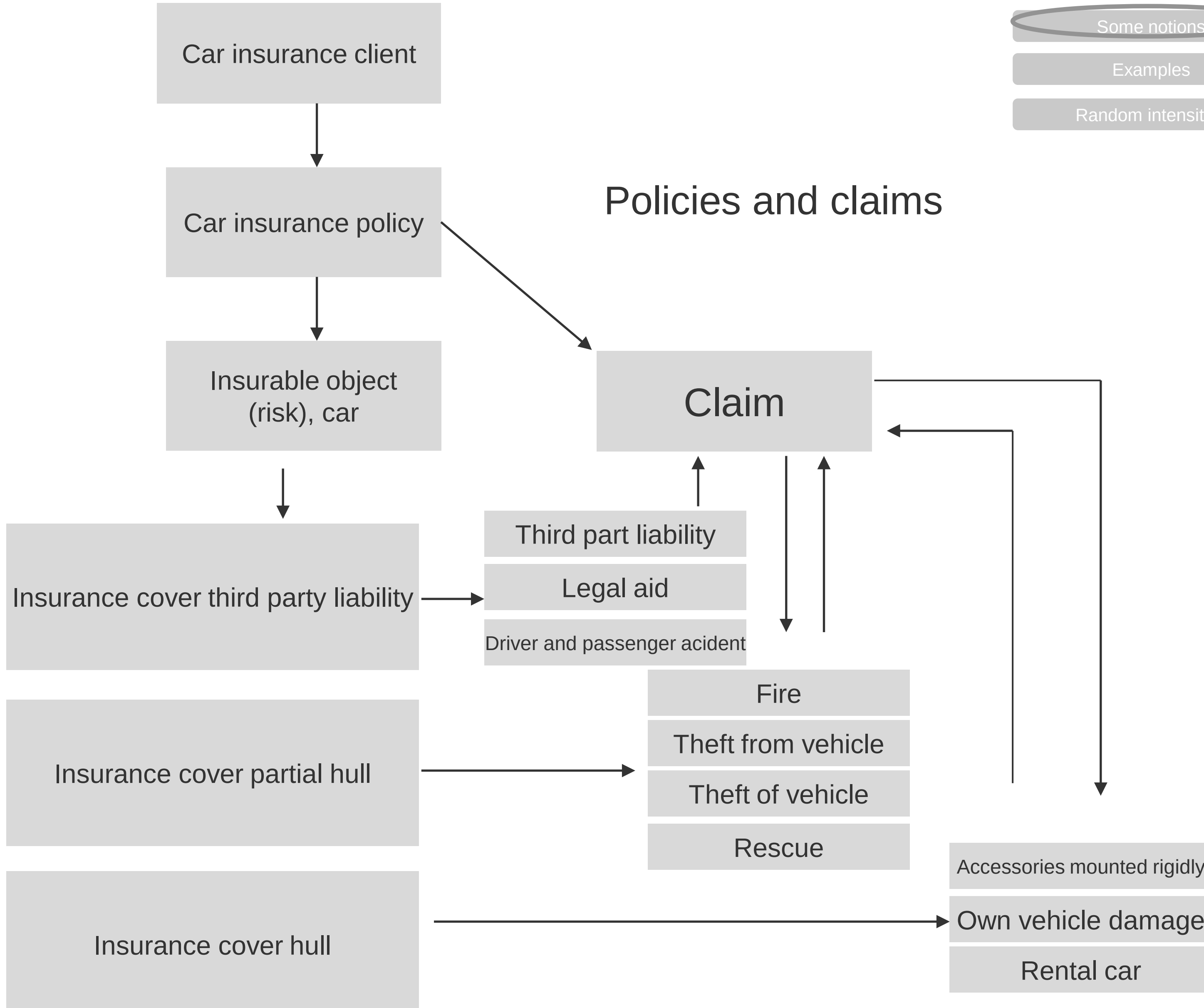
The sums of independent Poisson variables must remain Poisson, if $N_1,\ldots,N_J$ are independent and Poisson with parameters $\lambda_1,\ldots,\lambda_J$ then

$$\mathbf{N} = N_1 + \ldots + N_J \sim Poisson(\lambda_1 + \ldots + \lambda_J)$$

Policies and claims



Policies and claims



Some notes on the different insurance covers on the previous slide:

Third part liability is a mandatory cover dictated by Norwegian law that covers damages on third part vehicles, propterty and person. Some insurance companies provide additional coverage, as legal aid and driver and passenger accident insurance.

Partial Hull covers everything that the third part liability covers. In addition, partial hull covers damages on own vehicle caused by fire, glass rupture, theft and vandalism in association with theft. Partial hull also includes rescue. Partial hull does not cover damage on own vehicle caused by collision or landing in the ditch. Therefore, partial hull is a more affordable cover than the Hull cover. Partial hull also cover salvage, home transport and help associated with disruptions in production, accidents or disease.

Hull covers everything that partial hull covers. In addition, Hull covers damages on own vehicle in a collision, overturn, landing in a ditch or other sudden and unforeseen damage as for example fire, glass rupture, theft or vandalism. Hull may also be extended to cover rental car.

Some notes on some important concepts in insurance:

What is bonus?
Bonus is a reward for claim-free driving. For every claim-free year you obtain a reduction in the insurance premium in relation to the basis premium. This continues until 75% reduction is obtained.

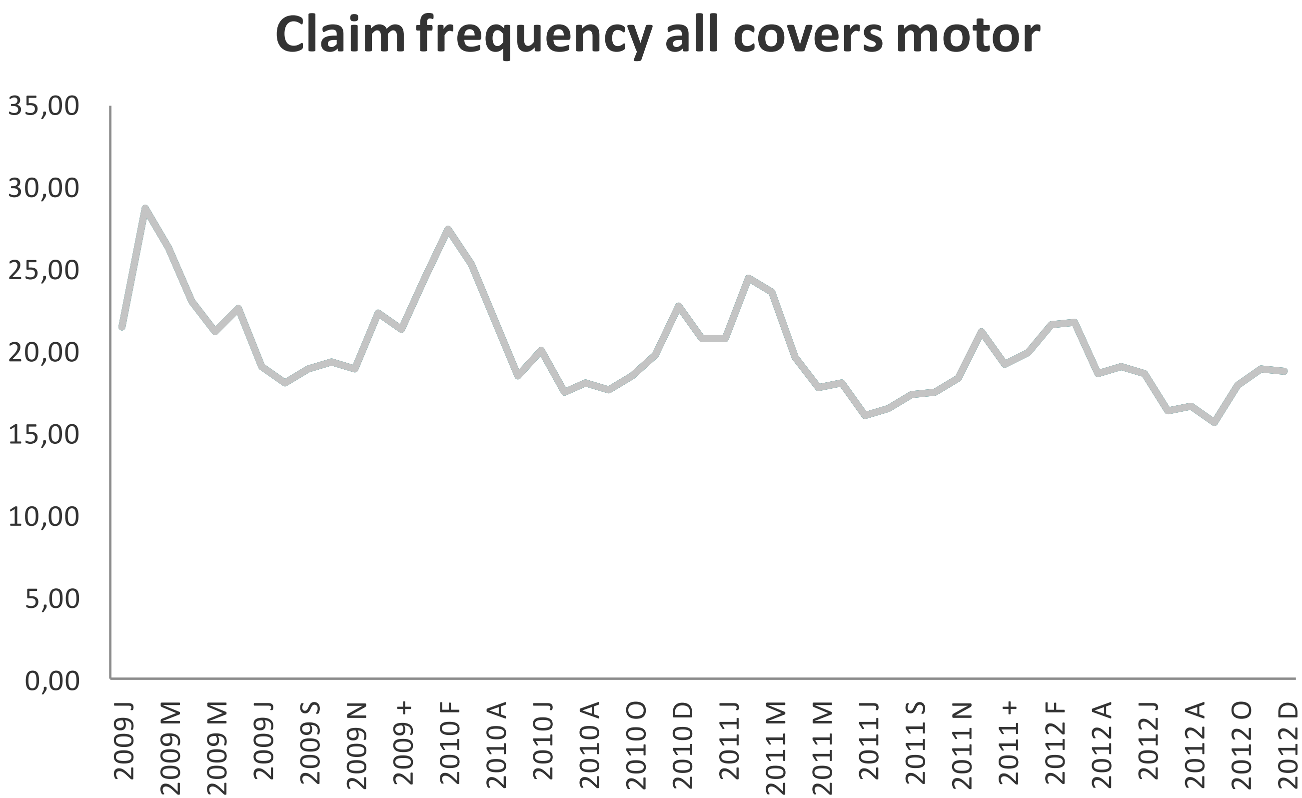
What is deductible?
The deductible is the amount the policy holder is responsible for when a claim occurs.

Does the deductible impact the insurance premium?
Yes, by selecting a higher deductible than the default deductible, the insurance premium may be significantly reduced. The higher deductible selected, the lower the insurance premium.

How is the deductible taken into account when a claim is disbursed?
The insurance company calculates the total claim amount caused by a damage entitled to disbursement. What you get from the insurance company is then the calculated total claim amount minus the selected deductible.

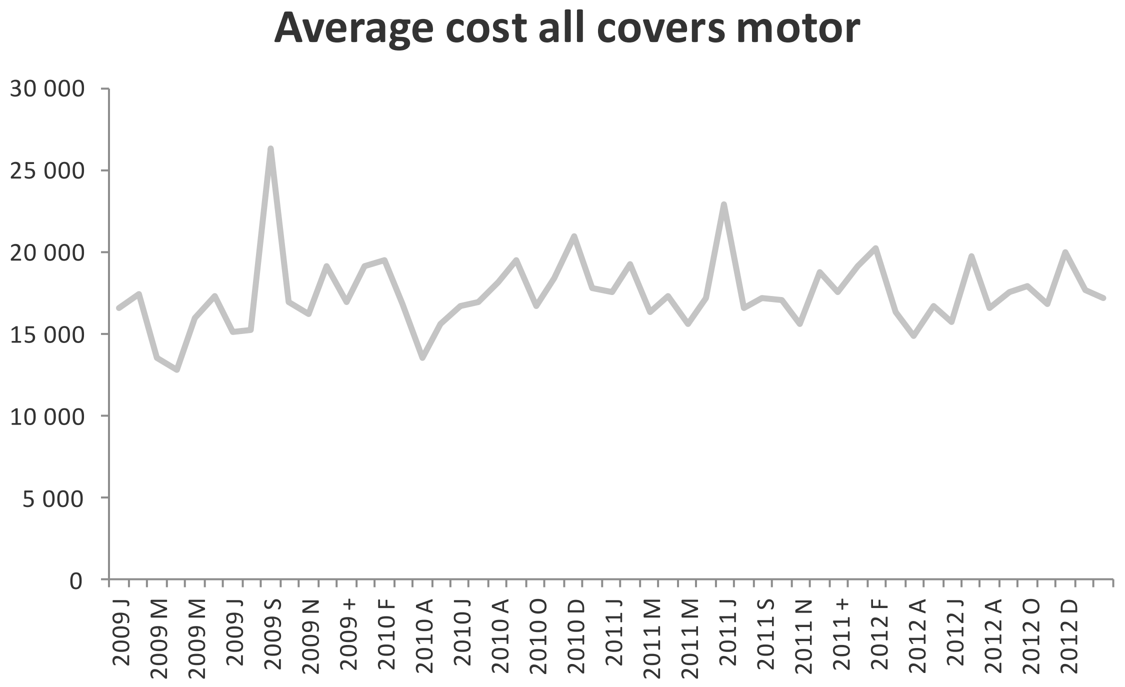
Key ratios – claim frequency

- The graph shows claim frequency for all covers for motor insurance
- Notice seasonal variations, due to changing weather condition throughout the years



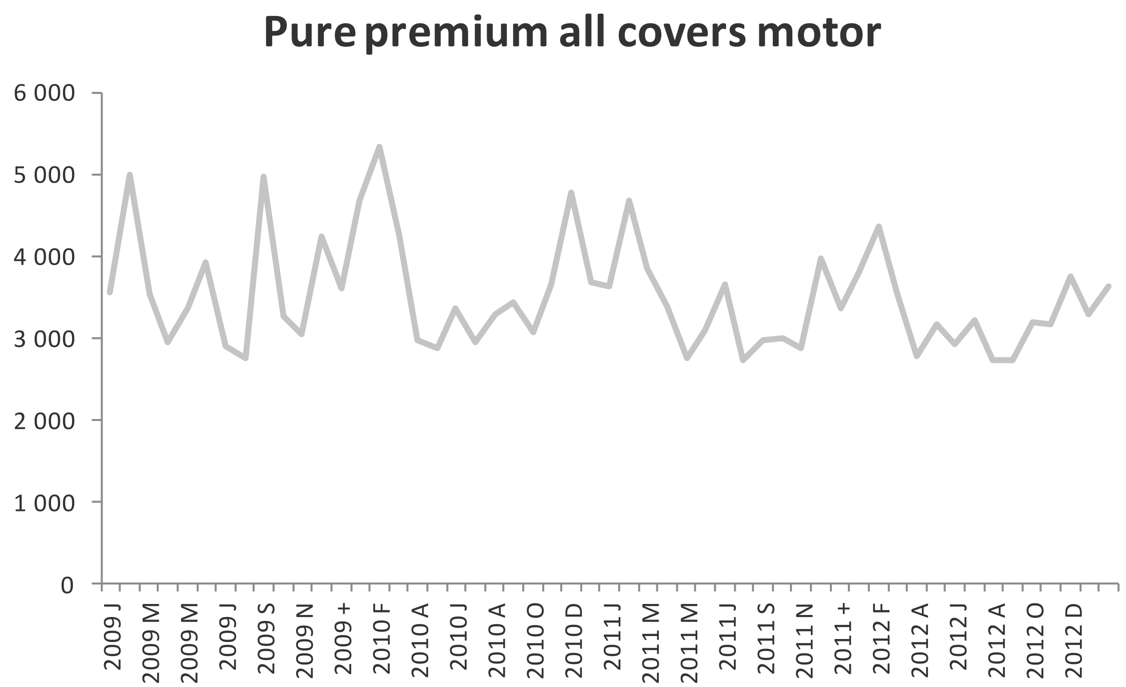
Key ratios – claim severity

- The graph shows claim severity for all covers for motor insurance



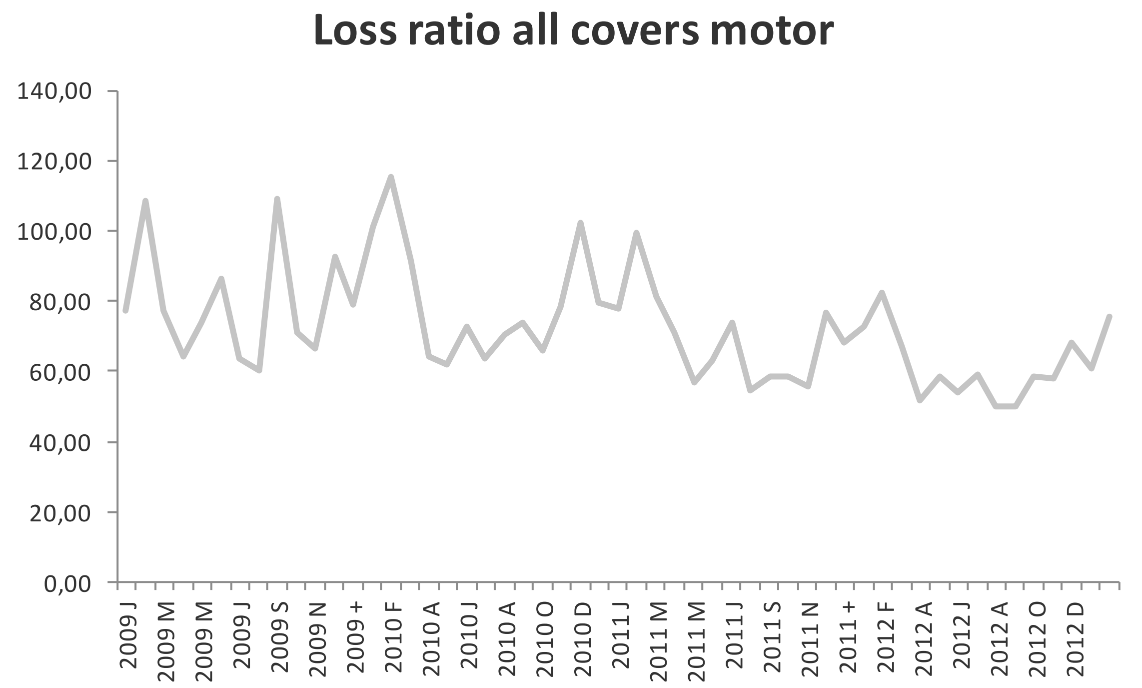
Key ratios – pure premium

- The graph shows pure premium for all covers for motor insurance



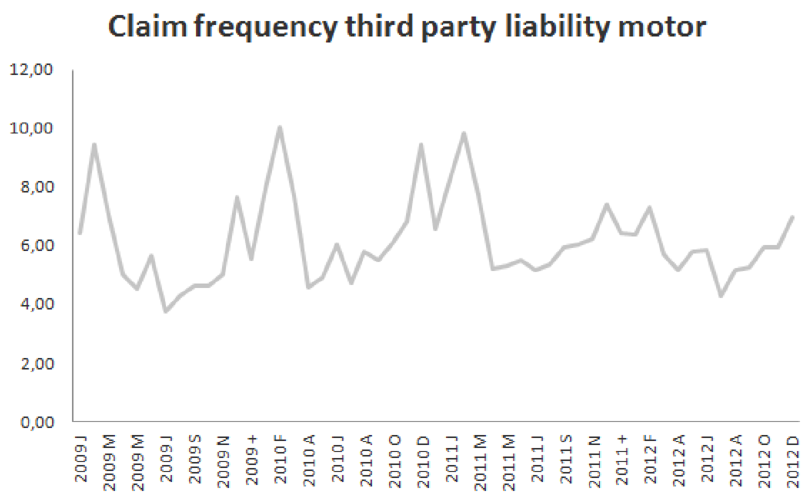
Key ratios – pure premium

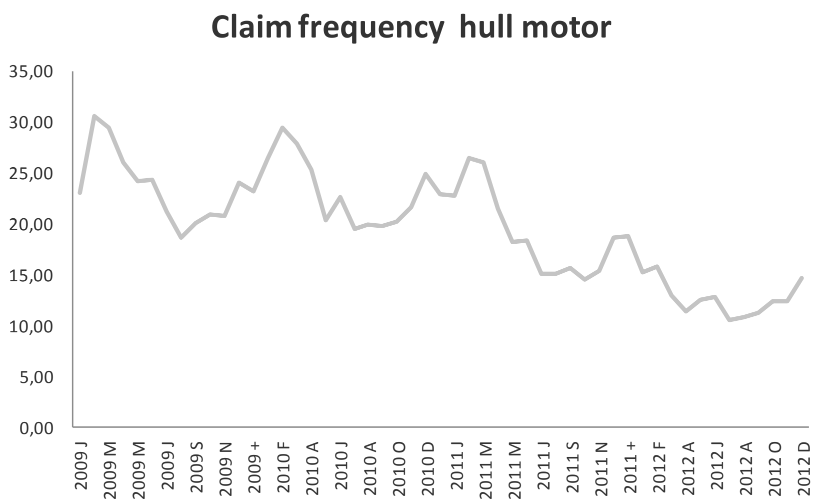
- The graph shows loss ratio for all covers for motor insurance



Key ratios – claim frequency TPL and hull

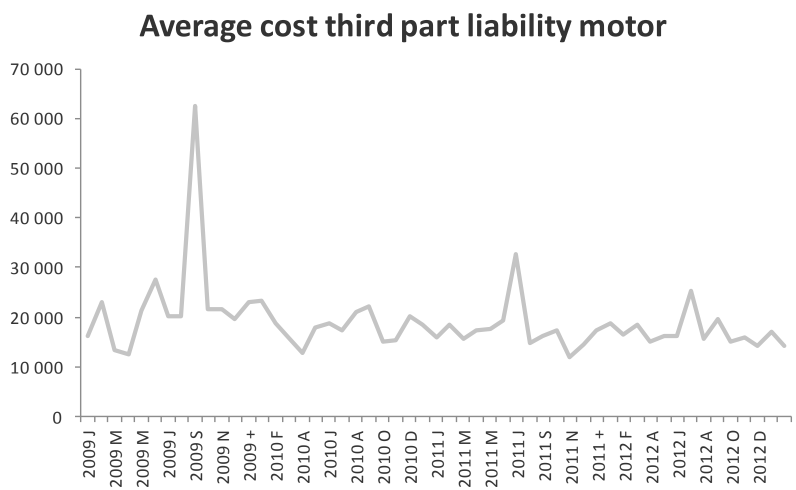
- The graph shows claim frequency for third part liability and hull for motor insurance

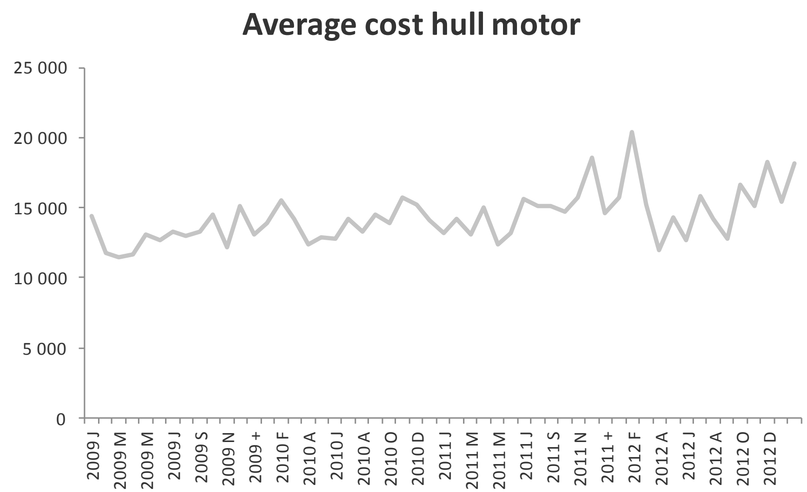




Key ratios – claim frequency and claim severity

- The graph shows claim severity for third part liability and hull for motor insurance





Random intensities (Chapter 8.3)

- How μ varies over the portfolio can partially be described by observables such as age or sex of the individual (treated in Chapter 8.4)
- There are however factors that have impact on the risk which the company can't know much about
  - Driver ability, personal risk averseness,
- This randomeness can be managed by making μ a stochastic variable
- This extension may serve to capture uncertainty affecting all policy holders jointly, as well, such as altering weather conditions
- The models are conditional ones of the form

$$N \mid \mu \sim Poisson(\mu T) \quad \text{and} \quad \mathbf{N} \mid \mu \sim Poisson(J\mu T)$$

Policy level   Portfolio level

- Let $\xi = E(\mu)$ and $\sigma = \text{sd}(\mu)$ and recall that $E(N \mid \mu) = \text{var}(N \mid \mu) = \mu T$

  which by double rules in Section 6.3 imply

$$E(N) = E(\mu T) = \xi T \quad \text{and} \quad \text{var}(N) = E(\mu T) + \text{var}(\mu T) = \xi T + \sigma^2 T^2$$

- Now E(N)<var(N) and N is no longer Poisson distributed

The rule of double variance

Let X and Y be arbitrary random variables for which

$$\xi(x) = E(Y \mid x) \quad \text{and} \quad \sigma^2 = \text{var}(Y \mid x)$$

Then we have the important identities

$$\underbrace{\xi = E(Y) = E\{\xi(X)\}}_{\text{Rule of double expectation}} \quad \text{and} \quad \underbrace{\text{var(Y)} = \text{E}\{\sigma^2(X)\} + \text{var}\{\xi(X)\}}_{\text{Rule of double variance}}$$

Recall rule of double expectation

$$E(E(Y \mid x)) = \int_{\text{all x}} (E(Y \mid x))\, f_X(x)dx = \int_{\text{all x}} \int_{\text{all y}} y f_{Y|X}(y|x)dy\, f_X(x)dx$$

$$= \int_{\text{all y}} \int_{\text{all x}} y f_{X,Y}(x,y)dxdy = \int_{\text{all y}} y \int_{\text{all x}} f_{X,Y}(x,y)dxdy = \int_{\text{all y}} y f_Y(y)dy = E(Y)$$

wikipedia tells us how the rule of double variance can be proved

Law of total variance

From Wikipedia, the free encyclopedia

In probability theory, the **law of total variance**[1] or **variance decomposition formula**, states that if X and Y are random variables on the same probability space, and the variance of Y is finite, then

$$\operatorname{Var}[Y] = \operatorname{E}(\operatorname{Var}[Y \mid X]) + \operatorname{Var}(\operatorname{E}[Y \mid X]).$$

Proof [edit source]

The law of total variance can be proved using the law of total expectation.[3] First,

$$\operatorname{Var}[Y] = \operatorname{E}[Y^2] - \operatorname{E}[Y]^2$$

from the definition of variance. Then we apply the law of total expectation to each term by conditioning on the random variable X:

$$= \operatorname{E}_X\left[\operatorname{E}[Y^2 \mid X]\right] - \operatorname{E}_X\left[\operatorname{E}[Y \mid X]\right]^2$$

Now we rewrite the conditional second moment of Y in terms of its variance and first moment:

$$= \operatorname{E}_X\left[\operatorname{Var}[Y \mid X] + \operatorname{E}[Y \mid X]^2\right] - \operatorname{E}_X[\operatorname{E}[Y \mid X]]^2$$

Since the expectation of a sum is the sum of expectations, the terms can now be regrouped:

$$= \operatorname{E}_X[\operatorname{Var}[Y \mid X]] + \left(\operatorname{E}_X\left[\operatorname{E}[Y \mid X]^2\right] - \operatorname{E}_X[\operatorname{E}[Y \mid X]]^2\right)$$

Finally, we recognize the terms in parentheses as the variance of the conditional expectation E[Y|X]:

$$= \operatorname{E}_X\left[\operatorname{Var}[Y \mid X]\right] + \operatorname{Var}_X\left[\operatorname{E}[Y \mid X]\right]$$

The rule of double variance

Var(Y) will now be proved from the rule of double expectation. Introduce

$$\hat{Y} = \xi(x) \quad \text{and note that} \quad \mathrm{E}(\hat{Y}) = \mathrm{E}(\mathrm{Y})$$

which is simply the rule of double expectation. Clearly

$$(Y-\xi)^2 = ((Y-\hat{Y}) + (\hat{Y}-\xi))^2 = (Y-\hat{Y})^2 + (\hat{Y}-\xi)^2 + 2(Y-\hat{Y})(\hat{Y}-\xi).$$

Passing expectations over this equality yields

$$\mathrm{var}(Y) = B_1 + B_2 + 2B_3$$

where

$$B_1 = E(Y-\hat{Y})^2, \quad B_2 = E(\hat{Y}-\xi)^2, \quad B_3 = E(Y-\hat{Y})(\hat{Y}-\xi),$$

which will be handled separately. First note that

$$\sigma^2(x) = E\{(Y-\xi(x))^2 \mid x\} = E\{(Y-\hat{Y})^2 \mid x\}$$

and by the rule of double expectation applied to $(Y-\hat{Y})^2$

$$E\{\sigma^2(x)\} = E\{(Y-\hat{Y})^2 = B_1.$$

The second term makes use of the fact that $\xi = E(\hat{Y})$ by the rule of double expectation so that

The rule of double variance

$$B_2 = \text{var}(\hat{Y}) = \text{var}\{\xi(x)\}.$$

The final term B_3 makes use of the rule of double expectation once again which yields

$$B_3 = E\{c(X)\}$$

where

$$c(X) = E\{(Y - \hat{Y})(\hat{Y} - \xi) \mid x\} = E\{(Y - \hat{Y}) \mid x\}(\hat{Y} - \xi)$$

$$= \{E(Y \mid x) - \hat{Y}\}(\hat{Y} - \xi) = \{\hat{Y} - \hat{Y}\}(\hat{Y} - \xi) = 0$$

And B_3=0. The second equality is true because the factor $(\hat{Y} - \xi)$ is fixed by X. Collecting the expression for B_1, B_2 and B_3 proves the double variance formula

Random intensities

Specific models for μ are handled through the mixing relationship

$$\Pr(N=n)=\int_0^\infty \Pr(N=n\mid\mu)g(\mu)d\mu \approx \sum_i \Pr(N=n\mid\mu_i)\Pr(\mu=\mu_i)$$

Gamma models are traditional choices for $g(\mu)$ and detailed below

Estimates of ξ and σ can be obtained from historical data without specifying $g(\mu)$. Let $n_1,\ldots,n_n$ be claims from n policy holders and $T_1,\ldots,T_J$ their exposure to risk. The intensity μ_j if individual j is then estimated as $\hat{\mu}_j = n_j / T_j$.

Uncertainty is huge. One solution is

$$\hat{\xi}=\sum_{j=1}^n w_j\hat{\mu}_j \quad \text{where} \quad w_j=\frac{T_j}{\sum_{i=1}^n T_i} \qquad (1.5)$$

and

$$\hat{\sigma}^2=\frac{\sum_{j=1}^n w_j(\hat{\mu}_j-\hat{\xi})^2-c}{1-\sum_{j=1}^n w_j^2} \quad \text{where} \quad c=\frac{(n-1)\hat{\xi}}{\sum_{i=1}^n T_i} \qquad (1.6)$$

Both estimates are unbiased. See Section 8.6 for details. 10.5 returns to this.

The negative binomial model

The most commonly applied model for muh is the Gamma distribution. It is then assumed that

$$\mu = \xi G \quad \text{where} \quad G \sim \text{Gamma}(\alpha)$$

Here $\text{Gamma}(\alpha)$ is the standard Gamma distribution with mean one, and μ fluctuates around ξ with uncertainty controlled by α. Specifically

$$E(\mu) = \xi \quad \text{and} \quad \text{sd}(\mu) = \xi / \sqrt{\alpha}$$

Since $\text{sd}(\mu) \to 0$ as $\alpha \to \infty$, the pure Poisson model with fixed intensity emerges in the limit.

The closed form of the density function of N is given by

$$\Pr(N = n) = \frac{\Gamma(n+\alpha)}{\Gamma(n+1)\Gamma(\alpha)} p^{\alpha}(1-p)^{n} \quad \text{where} \quad \text{p} = \frac{\alpha}{\alpha + \xi \text{T}}$$

for n=0,1,.... This is the negative binomial distribution to be denoted $\text{nbin}(\xi, \alpha)$.
Mean, standard deviation and skewness are

$$E(N) = \xi T, \quad sd(N) = \sqrt{\xi T(1 + \xi T / \alpha)}, \quad \text{skew(N)} = \frac{1 + 2\xi \text{T}/\alpha}{\sqrt{\xi T(1 + \xi T / \alpha)}} \qquad (1.9)$$

Where E(N) and sd(N) follow from (1.3) when $\sigma = \xi / \sqrt{\alpha}$ is inserted.
Note that if $N_1,...,N_J$ are iid then $N_1+...+N_J$ is nbin (convolution property).

Fitting the negative binomial

Moment estimation using (1.5) and (1.6) is simplest technically. The estimate of ξ is simply $\hat{\xi}$ in (1.5), and for α invoke (1.8) right which yields

$$\hat{\sigma} = \hat{\xi}/\sqrt{\hat{\alpha}} \quad \text{so that} \quad \hat{\alpha} = \hat{\xi}^2/\hat{\sigma}^2 .$$

If $\hat{\sigma} = 0$, interpret it as an infinite α or a pure Poisson model.

Likelihood estimation: the log likelihood function follows by inserting n_j for n in (1.9) and adding the logarithm for all j. This leads to the criterion

$$L(\xi,\alpha) = \sum_{j=1}^{n} \log(n_j + \alpha) - n\{\log(\Gamma(\alpha)) - \alpha\log(\alpha)\} +$$

$$\sum_{j=1}^{n} n_j \log(\xi) - (n_j + \alpha)\log(\alpha + \xi T_j)$$

where constant factors not depending on ξ and α have been omitted.