STK4540: Non-life Insurance Mathematics
First assignment

This assignment consists of two exercises, but the second one is only optional! Focus on Exercise 1 first. The deadline is Thursday 3rd of October at 14:30. To pass the assignment you just need to have **one** correct item out of the 12 items in Exercise 1. Pretty easy! But as a teacher I strongly encourage you to try to solve the whole assignment, including the optional exercise. Good luck!

Exercise 1

In this exercise we are interested in the law of the minimum and maximum of the claim sizes and the total aggregate claim size associated to an insurance portfolio. Consider a flow of i.i.d. claims $X_1, \dots, X_n, \dots$ with common distribution function F such that $F(0) = 0$, and a Poisson random variable N_λ independent of the claims with intensity $\lambda > 0$ describing the count number of claims.

(a) Consider the random variables $U_\lambda = \min\{X_1, \dots, X_{N_\lambda}\}$ if $N_\lambda > 0$ and $U_\lambda = 0$ and $V_\lambda = \max\{X_1, \dots, X_{N_\lambda}\}$ if $N_\lambda > 0$ and $V_\lambda = 0$ if $N_\lambda = 0$. Find the distribution function of U_λ and V_λ in terms of λ and F.

Assume for the rest of the exercise that we measure the claim sizes uniformly on the unit interval, i.e. the law of X_i is given by $\mathcal{L}_{X_i}(B) = P(\omega \in \Omega : X_i(\omega) \in B) = \mu([0,1] \cap B)$ for Borel sets B of $\mathbb{R}$ where μ denotes the Lebesgue measure. In particular, $F_{X_i}(x) = \mathcal{L}_{X_i}((-\infty, x]) = x\mathbf{1}_{\{0 \leqslant x < 1\}} + \mathbf{1}_{\{x \geqslant 1\}}$.

(b) Show that both the sequence $\{\lambda(1 - V_\lambda)\}_{\lambda > 0}$ and $\{\lambda U_\lambda\}_{\lambda > 0}$ converge in distribution to an exponential distribution with rate 1. Show that they do not converge in probability.

(c) Make a short program simulating values of U_λ and V_λ for some λ of your choice (try small and big) and compare the simulated distribution with that of an exponential distribution with rate 1.

(d) For any $\alpha \in (0,1)$, construct an approximate $(1 - \alpha)100\%$-confidence interval for λ.

(e) Imagine you have a consistent estimator $\widehat{\lambda}$ of λ at your disposal. For any $\alpha \in (0,1)$, construct an approximate $(1 - \alpha)100\%$-confidence interval for the maximum claim size. If for instance $\alpha = 0.05$ and you observe $\widehat{\lambda_{obs}} = 100$, what do you expect the maximum claim size to be? Is this reasonable? If then $\widehat{\lambda}_{obs} = 1000$, what happens with V_λ?

(f) Denote by S the total aggregate claim amount $S = \sum_{i=1}^{N_\lambda} X_i$. Argue that for every $\varepsilon > 0$ we have

$$P(N_\lambda U_\lambda > \varepsilon) \leqslant P(S > \varepsilon) \leqslant P(N_\lambda V_\lambda > \varepsilon).$$

(g) Using (a) and (f) prove that for every $\varepsilon > 0$ the tail distribution of S can be bounded by

$$\sum_{n=\lfloor\varepsilon\rfloor+1}^{\infty} \left(e^{-\frac{\varepsilon}{n}\lambda} - e^{-\lambda}\right) e^{-\lambda}\frac{\lambda^n}{n!} \leqslant P(S > \varepsilon) \leqslant \sum_{n=\lfloor\varepsilon\rfloor+1}^{\infty} \left(1 - e^{\left(\frac{\varepsilon}{n}-1\right)\lambda}\right) e^{-\lambda}\frac{\lambda^n}{n!},$$

where $\lfloor\varepsilon\rfloor = \max\{m \in \mathbb{N} : m \leqslant \varepsilon\}$ denotes the integer part of ε and deduce that $\lim_{\varepsilon\to 0} P(S > \varepsilon) = 1 - e^{-\lambda}$ and $\lim_{\varepsilon\to\infty} P(S > \varepsilon) = 0$.

(h) A *stop-loss* reinsurance contract with retention level $K > 0$ is a contract where the reinsurer covers losses in the portfolio exceeding a well-defined limit K, the so-called *ceding company's retention level.* This means that the reinsurer pays for $R_{SL} = (S-K)_+$, where $x_+ = \max\{x, 0\}$. This type of insurance is useful for protecting the company against insolvency due to excessive claims on the coverage. The net value is then given by $p(K) = E[R_{SL}] = E[(S-K)_+]$. Argue that

$$p(K) = E[R_{SL}] = \int_K^\infty P(S > \varepsilon)d\varepsilon.$$

The rest of the exercise will deal with finding bounds for p in terms of K. For this purpose it may be useful to recall the following analytical inequalities; for every $x, y \in \mathbb{R}$ such that $x < y$ we have

$$e^x(y-x) \leqslant e^y - e^x \leqslant e^y(y-x).$$

Also for $r \in \mathbb{N}$ and $\lambda > 0$ we have

$$\frac{\lambda^{r+1}}{(r+1)!} \leqslant \sum_{n=r+1}^{\infty} \frac{\lambda^n}{n!} \leqslant e^\lambda \frac{\lambda^{r+1}}{(r+1)!}.$$

(i) Using the information obtained so far, show that

$$e^{-\lambda} f(\varepsilon) \leqslant P(S > \varepsilon) \leqslant f(\varepsilon),$$

where

$$f(\varepsilon) = \lambda e^{-\lambda} \sum_{n=\lfloor\varepsilon\rfloor+1}^{\infty} \left(1 - \frac{\varepsilon}{n}\right) \frac{\lambda^n}{n!}.$$

(j) Argue that

$$\lambda e^{-\lambda}\left(1 - \frac{\varepsilon}{\lfloor\varepsilon\rfloor + 1}\right) \frac{\lambda^{\lfloor\varepsilon\rfloor+1}}{(\lfloor\varepsilon\rfloor+1)!} \leqslant f(\varepsilon) \leqslant \lambda \frac{\lambda^{\lfloor\varepsilon\rfloor+1}}{(\lfloor\varepsilon\rfloor+1)!},$$

and plot upper bounds for $P(S > \varepsilon)$ for $\varepsilon > 0$ for different values of λ, say $\lambda \in \{1, 5, 10, 20, 50, 100\}$. Find ε such that $P(S > \varepsilon) \approx 0.01$. Such value is referred to as *reserve*.

(k) Assume that K is a positive integer number (this is fine since K is usually given in terms of a currency). Show that the (fair) price of a stop-loss contract can be bounded by

$$p(K) \leqslant \lambda e^{\lambda} \frac{\lambda^{K+1}}{(K+1)!}.$$

Hint: At some point you need to compute the integral of $\frac{\lambda^{\lfloor \varepsilon \rfloor + 1}}{(\lfloor \varepsilon \rfloor + 1)!}$ w.r.t. ε along $[K, \infty]$. Chop up the interval $[K, \infty)$ in pieces of length one and then sum up. Lastly, use the bounds provided in item (h).

(l) Plot the bounds as a function of K for different values of λ, say $\lambda \in \{1, 5, 10, 20, 50, 100\}$.

Exercise 2 (optional!)

We consider now the setting of previous exercise (you can use item (a)) for the case where the claim sizes are discrete and uniformly distributed on $\{0, 1, \ldots, M\}$ for a fixed integer level $M \geqslant 1$, i.e. the law of X_i is given by $\mathcal{L}_{X_i}(B) = P(\omega \in \Omega : X_i(\omega) \in B) = \frac{1}{M+1} \#(\{0, 1, \ldots, M\} \cap B)$ for subsets B of $\mathbb{N}$ where $\#$ denotes the counting measure. In particular, $P(X_i = k) = \mathcal{L}_{X_i}(\{k\}) = \frac{1}{M+1}$ for all $k = 0, \ldots, M$ also known as discrete uniform distribution.

(a) Define V_λ as before. Show that the distribution function of V_λ is given by

$$P(V_\lambda \leqslant k) = e^{-\lambda \frac{M-k}{M+1}}, \quad k = 0, \ldots, M.$$

(b) Consider the sequence $\lambda_n = n$, $n \geqslant 1$. Show that V_{λ_n} converges to M almost surely as $n \to \infty$. Hint: Use the first Borel-Cantelli lemma.

(c) Find an expression for exact probabilities $p_m = P(S = m)$ for each integer $m \geqslant 1$ using Panjer's recursion scheme. You should obtain

$$p_m = \frac{\lambda}{M+1} \frac{1}{m} \sum_{i=1}^{M} i p_{m-i}, \quad m \geqslant M,$$

given $p_0 = e^{-\lambda \frac{M}{M+1}}$ and $p_k = 0$ for $k < 0$.

(d) Assume for a moment that $M = 1$ (Bernoulli case). Find the exact distribution of S with support on $\mathbb{N}$ and find $E[S]$ and $Var[S]$.

(e) Using the previous exercise in the case $M = 1$ we can prove by CLT that

$$\frac{S - \lambda/2}{\sqrt{\lambda/2}} \to N(0, 1)$$

in law, where $N(0, 1)$ denotes a standard normal distribution. Plot the different distributions of the above random variable for say $\lambda \in \{1, 5, 10, 20, 50, 100, \}$ and compare to the standard normal.

(f) For $\lambda = 10$ and $M = 1$ compute $P(S \geqslant 5)$ and $P(S \geqslant 10)$ using the exact distribution from (d) and the normal approximation from (e). What do you observe? Comment.