STK4540: Non-life Insurance Mathematics

Exercise list 0

Exercise 1

We say that a random variable X has a Pareto distribution with scale parameter $c \geqslant 0$ and shape parameter $\theta > 0$ if it is absolutely continuous with density function given by

$$f(x) = \frac{\theta c^\theta}{x^{\theta+1}}, \quad x > c.$$

(a) Check that f is indeed a density function.

(b) Compute the mean and the variance.

(c) Find estimators $\widehat{c}_n$ and $\widehat{\theta}_n$ for c and θ, respectively, based on a random sample $X_1, \ldots, X_n$.

(d) What is the distribution of $X|X > d$ where $d > c$?

(e) Let U be a uniformly distributed random variable on $(0, 1]$. Prove that

$$\frac{c}{U^{1/\theta}}$$

is Pareto distributed.

Exercise 2

Let X denote the proportion of allotted time that a randomly selected student spends working on a certain aptitude test. Suppose the pdf of X is

$$f_\theta(x) = (\theta + 1)x^\theta, \quad 0 < x < 1,$$

where $\theta > -1$. A random sample of ten students yields data $x_1 = 0.92$, $x_2 = 0.79$, $x_3 = 0.9$, $x_4 = 0.65$, $x_5 = 0.86$, $x_6 = 0.47$, $x_7 = 0.73$, $x_8 = 0.97$, $x_9 = 0.94$, $x_{10} = 0.77$.

(a) Use the method of moments to obtain an estimator of θ, and then compute the estimate for this data.

(b) Obtain the maximum likelihood estimator of θ, and then compute the estimate for the given data.

(c) Run a simulation to find out which of the two estimator has lowest variance.

Exercise 3

For $(x, y) \in \mathbb{R}^2$,

$$f(x, y) = \frac{4y}{x^3} \mathbf{1}_{\{0<x<1, 0<y<x^2\}}.$$

(a) Verify that f is indeed a probability density function, then compute the marginal densities f_X and f_Y.

(b) Compute $f_{Y|X}(y|x)$ and deduce that

$$E[Y|X] = \frac{2}{3} X^2.$$

(c) Establish that

$$f_{X|Y}(x|y) = \frac{2y}{1-y} \frac{1}{x^3} \mathbf{1}_{\{0<x<1, 0<y<x^2\}}$$

and deduce $E[X|Y]$.

Exercise 4

Roll a die until you get a 6. Let Y be the total number of rolls and X the total number of 1's we get. Compute $E[X|Y]$.

Exercise 5

Let X_i, $i = 1, \ldots, n$ be independent Poisson distributed random variables each with intensity parameter $\lambda_i > 0$, $i = 1, \ldots, n$ respectively. Show that the sum $Y = \sum_{i=1}^n X_i$ is also Poisson distributed with intensity $\lambda = \sum_{i=1}^n \lambda$.

Exercise 6

Supose X and Y are square-integrable random variables (i.e. with finite second order moment) and that $E[X|Y] = Y$ and $E[Y|X] = X$. Show that $X = Y$ almost surely.

Exercise 7

Let $X_1, \ldots, X_n$ be a random sample from a gamma distribution with parameters α and β, i.e. the common density function is given by

$$f_{\alpha,\beta}(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x > 0.$$

(a) Derive the equations whose solution yields the maximum likelihood estimators of α and β. Do you think they can be solved explicitly?

(b) Show that the mle of $\mu = \alpha/\beta$ is $\widehat{\mu}_{mle} = \overline{X}_n$.

Exercise 8

Let $\{X_n\}$ be a sequence of random variables with probabilities $P(X_n = 1) = \frac{1}{n}$ and $P(X_n = 0) = 1 - \frac{1}{n}$. Prove that X_n converges to 0 in probability but not almost surely.

Exercise 9

Study the different types of convergence of the sequence of independent random variables $\{X_n\}_{n \geqslant 1}$ with law $P(X_n = 0) = 1 - \frac{1}{n}$ and $P(X_n = n) = \frac{1}{n}$.

Exercise 10

Consider a sequence of independent identically distributed random variables $\{U_n\}_{n \geqslant 1}$ such that $P(U_n = 1) = p$, $P(U_n = -1) = q$ with $p + q = 1$ and $0 \leqslant p \leqslant 1$. We construct the sequence:

$$V_1 = U_1, \quad V_2 = U_1 U_2, \quad \dots \quad V_n = \prod_{k=1}^{n} U_k.$$

(a) Compute $E[V_n]$.

(b) Find p_n and q_n where $p_n = P(V_n = 1)$ and $q_n = P(V_n = -1)$.

(c) Analyse the convergence in distribution of the sequence $\{V_n\}_{n \geqslant 1}$.

Exercise 11

Consider a sequence of independent Bernoulli distributed random variables with parameter p_n for each $n \geqslant 1$ (i.e. probability of obtaining 1). For each natural number $n \geqslant 1$ let T_n denote the first time this sequence takes the value 1.

(a) What is the law of T_n?

(b) Assume $p_n = \frac{\lambda}{n}$, $\lambda > 0$ and define $S_n = \frac{T_n}{n}$. Compute $E[S_n]$.

(c) Prove that the sequence $\{S_n\}_{n \geqslant 1}$ convergences in law to an exponential distribution with parameter λ.

Exercise 12

Consider an i.i.d. sequence of random variables $\{X_n\}_{n \geqslant 1}$ with density

$$f(x) = \exp(\lambda - x)\, \mathbf{1}_{(\lambda, \infty)}(x).$$

Consider now the sequence of means $Z_n = \frac{\sum_{j=1}^{n} X_j}{n}$. Show that

(a) the sequence $\{Z_n\}_{n \geqslant 1}$ converges in L^2 and in probability to $1 + \lambda$.

(b) the minimum $\min\{X_1, \dots, X_n\}$ converges in probability and almost surely to λ.

Exercise 13

Consider a sequence $\{X_n\}_{n\geqslant 1}$ of independent uniformly distributed random variables on the interval $[-1,1]$. We define

$$M_n = \max\{X_1 \ldots, X_n\},$$
$$N_n = \min\{X_1 \ldots, X_n\}.$$

(a) Compute the distribution functions of M_n and N_n.

(b) Analyse the different types of convergence: in probability, L^2 and almost surely.

Exercise 14

Let $X_1, \ldots, X_n$ be i.i.d. following a uniform distribution on $[0,1]$. Consider a random variable Y_m independent of the sequence with a Poisson distribution with parameter $m \in \mathbb{R}$, $m > 0$. We define

$$V_m = \max\{X_1, \ldots, X_{Y_m}\} \text{ if } Y_m > 0,$$

and $V_m = 0$ if $Y_m = 0$.

(a) Find the distribution function and density function of V_m.

(b) Compute $\lim_{m\to\infty} E[V_m]$.

(c) Show that the sequence $\{m(1-V_m)\}_{m>0}$ converges in distribution when $m \to \infty$ and find the limiting distribution.