STK4540: Non-life Insurance Mathematics

Exercise list 3

Exercise 1

($E[Z_1] = 0$ in the renewal model leads to unavoidable ruin with probability 1)

We know that the ruin probability $\psi(u)$ in the renewal model has representation

$$\psi(u) = P\left(\sup_{n \geqslant 1} S_n > u\right),$$

where $S_n = Z_1 + \cdots + Z_n$ is a random walk with iid step sizes $Z_i = X_i - cW_i$, $i \geqslant 1$. Assume the conditions $E[Z_1] = 0$ and $Var[Z_1] < \infty$ hold.

(a) Apply the central limit theorem to show that

$$\lim_{u \to +\infty} \psi(u) \geqslant 1 - \phi(0) = 0.5,$$

where ϕ is the standard normal distribution function. Hint: Notice that $\psi(u) \geqslant P(S_n > u)$ for every $n \geqslant 1$.

(b) It turns out that one can show a much finer estimate for the tail probability (we will not show this). A standard *Brownian motion* $\{B_t\}_{t \geqslant 0}$ is a stochastic process with independent stationary increments and continuous sample paths, starts at zero, i.e. $B_0 = 0$ a.s., and $B_t \sim N(0, t)$ for $t \geqslant 0$. One can show that

$$\psi(u) \geqslant P\left(\max_{0 \leqslant s \leqslant 1} B_s \geqslant 0\right).$$

It is a well-known fact that Brownian motion satifies the reflection principle

$$P\left(\max_{0 \leqslant s \leqslant 1} B_s \geqslant x\right) = 2P(B_1 \geqslant x), \quad x \geqslant 0.$$

Use the latter result to show that

$$\lim_{u \to +\infty} \psi(u) = 1.$$

(c) Conclude from (c) that $\psi(u) = 1$ for every $u > 0$.

Exercise 2

Consider the total claim amount process

$$S(t) = \sum_{i=1}^{N(t)} X_i, \quad t \geqslant 0,$$

where $\{X_i\}_i$ are iid positive claim sizes, independent of the Poisson process N with an a.e. positive and continuous intensity function λ. Choose the premium such that

$$p(t) = c \int_0^t \lambda(s) ds = c\mu(t),$$

for some premium rate $c > 0$ and consider the ruin probability

$$\psi(u) = P\left(\inf_{t \geqslant 0} (u + p(t) - S(t)) < 0 \right),$$

for some positive initial capital u. Show that $\psi(u)$ coincides with the ruin probability in the Cramér-Lundberg model with Poisson intensity 1, initial capital u and premium rate c. Which condition is needed in order to avoid ruin with probability 1?

Exercise 3

Consider the Cramér-Lundberg model with Poisson intensity λ and chi-squared χ_4^2 distributed claim sizes X_i with 4 degrees of freedom, i.e. with density $f(x) = \frac{x}{4} e^{-x/2}$, $x > 0$.

(a) Calculate the moment generating function $m_{X_1}(h)$ of X_1. For which $h \in \mathbb{R}$ is the function well-defined?

(b) Derive the NPC.

(c) Calculate the adjustment coefficient under the NPC.

(d) Try to find an expression for the survival probability $\varphi(u)$ and the ruin probability $\Psi(u)$, as close as you can.

Exercise 4

Consider the Cramér-Lundberg model with Poisson intensity λ and $\Gamma(\gamma, \beta)$ distributed claim sizes X_i with density $f(x) = \frac{\beta^\gamma}{\Gamma(\gamma)} x^{\gamma-1} e^{-\beta x}$, $x > 0$.

(a) Calculate the moment generating function $m_{X_1}(h)$ of X_1. For which $h \in \mathbb{R}$ is the function well-defined?

(b) Derive the NPC.

(c) Calculate the adjustment coefficient under the NPC.

(d) Assume the claim sizes are $\Gamma(n, \beta)$ distributed for some integer $n \geqslant 1$. Write $\psi^{(n)}(u)$ for the corresponding ruin probability with initial capital $u > 0$. Suppose that the same premium $p(t) = ct$ is charged for $\Gamma(n, \beta)$ and $\Gamma(n+1, \beta)$ distributed claim sizes. Show that $\psi^{(n)}(u) \leqslant \psi^{(n+1)}(u)$, $u > 0$.