STK4540: Non-life Insurance Mathematics

Exercise list 4

Exercise 1

Consider the risk process $U(t) = u + ct - S(t)$ in the Cramér-Lundberg model.

(a) Show that $S(s) = \sum_{i=1}^{N(s)} X_i$ is independent of $S(t) - S(s)$ for $s < t$.

(b) Use (a) to calculate

$$E\left[e^{-hU(t)}\middle|S(s)\right]$$

for $s < t$ and some $h > 0$ Here we assume that $Ee^{hS(t)}$ is finite. Under the assumption that the Lundberg coefficient r exists show the following relation:

$$E\left[e^{-rU(t)}\middle|S(s)\right] = e^{-rU(s)} \quad a.s.$$

(c) Under the assumptions of (b) show that $Ee^{-rU(t)}$ does not depend on t.

Exercise 2

Consider the risk process with premium rate c in the Cramér-Lundberg model with Poisson intensity λ. Assume that the adjustment coefficient r exists as the unique solution to the equation $1 = E[e^{r(X_1 - cW_1)}]$. Write $m_A(h)$ for the moment generating function of any random variable A and $\rho = c/(\lambda EX_1) - 1 > 0$ for the safety loading. Show that r can be determined as the solution to each of the following equations

$$\begin{aligned}
\lambda + cr &= \lambda m_{X_1}(r), \\
0 &= \int_0^\infty [e^{rx} - (1+\rho)]P(X_1 > x)]dx, \\
e^{cr} &= m_{S(1)}(r), \\
c &= \frac{1}{r}\log m_{S(1)}(r).
\end{aligned}$$

Exercise 3

Assume the Cramér-Lundberg model with the Net Profit Condition (NPC). We also suppose that the moment generating function $m_{X_1}(h) = Ee^{hX_1}$ of the claim sizes X_i is finite for all $h > 0$. Show that there exists a unique solution $r > 0$ (Lundberg coefficient) to the equation $1 = Ee^{h(X_1 - cW_1)}$.

Exercise 4

Consider the risk process $U(t) = u + ct - S(t)$ with total claim amount $S(t) = \sum_{i=1}^{N(t)} X_i$, where the iid sequence $\{X_i\}_{i\geqslant 1}$ of $Exp(\gamma)$ distributed claim sizes is independent of the mixed homogeneous Poisson process N. In particular, we assume

$$\{N(t)\}_{t\geqslant 0} = \{\widetilde{N}(\theta t)\}_{t\geqslant 0}$$

where $\widetilde{N}$ is a standard homogeneous Poisson process, independent of the positive mixing variable θ.

(a) Conditionally on θ, determine the NPC and the probability of ruin for this model,i.e.,

$$P\left(\inf_{t\geqslant 0} U(t) < 0 \,\middle|\, \theta\right).$$

(b) Apply the results of part (a) to determine the ruin probability

$$\psi(u) = P\left(\inf_{t\geqslant 0} U(t) < 0\right).$$

(c) Use part (b) to give conditions under which $\psi(u)$ decays exponentially fast to zero as $u \to \infty$.

(d) What changes in the above calculations if you choose the premium $p(t) = (1+\rho)(\theta/\gamma)t$ for some $\rho > 0$? This means that you consider the risk process $U(t) = u + p(t) - S(t)$ with random premium adjusted to θ.

Exercise 5

Consider a reinsurance company with risk process $U(t) = u + ct - S(t)$,where the total claim amount $S(t) = \sum_{i=1}^{N(t)} (X_i - x)_+$ corresponds to an excess-of-loss treaty. Moreover, N is homogeneous Poisson with intensity λ, independent of the iid sequence $\{X_i\}_{i\geqslant 1}$ of $Exp(\gamma)$ random variables. We choose the premium rate according to the expected value principle:

$$c = (1+\rho)\lambda E[(X_1 - x)_+]$$

for some positive safety loading ρ.

(a) Show that $c = (1+\rho)\lambda e^{-\gamma x}/\gamma$.

(b) Show that

$$\phi_{(X_1-x)_+}(t) = E[e^{it(X_1-x)_+}] = 1 + \frac{it}{\gamma - it} e^{-x\gamma}, \quad t \in \mathbb{R}.$$

(c) Show that $S(t)$ has the same distribution as $\widehat{S}(t) = \sum_{i=1}^{\widehat{N}(t)} X_i$, where $\widehat{N}$ is a homogeneous Poisson process with intensity $\widehat{\lambda} = \lambda e^{-x\gamma}$, independent of $\{X_i\}_i$.

(d) Show that the processes S and $\widehat{S}$ have the same finite-dimensional distributions. Hint: The compound Poisson processes S and $\widehat{S}$ have independent and stationary increments. Use (c).

(e) Define the risk process $\widehat{U}(t) = u + ct - \widehat{S}(t)$, $t \geqslant 0$. Show that

$$\psi(u) = P\left(\inf_{t \geqslant 0} U(t) < 0\right) = P\left(\inf_{t \geqslant 0} \widehat{U}(t) < 0\right)$$

and calculate $\psi(u)$. Hint: Use (d).