STK4540: Non-life Insurance Mathematics

Exercise list 6

Exercise 1

Assume the heterogeneity model and consider the ith policy. We suppress the dependence on i in the notation. Given $\theta > 0$, let the claim sizes $X_1, \ldots, X_n$ in the policy be iid Pareto-distributed with parameters (λ, θ), i.e.,

$$\overline{F}(x|\theta) = P(X_i > x|\theta) = \left(\frac{\lambda}{x}\right)^{\theta}, \quad x > \lambda.$$

Assume that θ is $\Gamma(\gamma, \beta)$ distributed with density

$$f_{\gamma,\beta}(y) = \frac{\beta^{\gamma}}{\Gamma(\gamma)} y^{\gamma-1} e^{-\beta y}, \quad y > 0.$$

(a) Show that $\theta|\vec{X}$ with $\vec{X} = (X_1, \ldots, X_n)'$ is $\Gamma(\overline{\gamma}, \overline{\beta})$ distributed with parameters

$$\overline{\gamma} = \gamma + n, \quad \overline{\beta} = \beta + \sum_{i=1}^{n} \log(X_i/\lambda).$$

(b) A reinsurance company takes into account only the values X_i exceeding a known high threshold K. They "observe" the counting variables $Y_i = I_{(K,\infty)}(X_i)$ for a known threshold $K > \lambda$. The company is interested in estimating $P(X_1 > K|\theta)$.

 (i) Give a naive estimator of $P(X_1 > K|\theta)$ based on the empirical distribution function of $X_1, \ldots, X_n$.

 (ii) Determine the a.s. limit of this estimator as $n \to \infty$. Does it coincide with $P(X_1 > K|\theta)$?

(c) Show that Y_i, given θ, is $Bin(1, p(\theta))$ distributed, where $p(\theta) = E[Y_1|\theta]$. Compare $p(\theta)$ with the limit in (b), (ii).

(d) Show that the Bayes estimator of $p(\theta) = E[Y_1|\theta]$ based on the data $X_1, \ldots, X_n$ is given by

$$\frac{(\beta + \sum_{i=1}^{n} \log(X_i/\lambda))^{\gamma+n}}{(\beta + \sum_{i=1}^{n} \log(X_i/\lambda) + \log(K/\lambda))^{\gamma+n}}.$$

Exercise 2

Assume the heterogeneity model and consider a policy with one observed claim number X and corresponding heterogeneity parameter θ. We assume that $X|\theta$ is $Pois(\theta)$-distributed, where θ has a continuous density f_θ on $(0,\infty)$. Notice that $E[X|\theta]=\theta$.

(a) Determine the conditional density $f_\theta(y|X=k)$, $k=0,1,\dots$, of $\theta|X$ and use this information to calculate the Bayes estimator $m_k=E[\theta|X=k]$, $k=0,1,2,\dots$

(b) Show that

$$m_k=(k+1)\frac{P(X=k+1)}{P(X=k)},\quad k=0,1,\dots$$

(c) Show that

$$E[\theta^l|X=k]=\prod_{i=0}^{l-1}m_{k+i},\quad k\geqslant 0,l\geqslant 1.$$

Exercise 3

Consider the ith policy in a heterogeneity model. We suppress the dependence on i in the notation.. We assume the heterogeneity parameter θ to be $Beta(a,b)$-distributed with density

$$f_\theta(y)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}y^{a-1}(1-y)^{b-1},\quad 0<y<1,\quad a,b>0.$$

Given θ, the claim numbers $X_1,\dots,X_n$ are iid $Bin(k,\theta)$-distributed.

(a) Calculate the conditional density $f_\theta(y|\vec{X}=\vec{x})$ of θ given

$$\vec{X}=(X_1,\dots,X_n)'=\vec{x}=(x_1,\dots,x_n)'.$$

(b) Calculate the Bayes estimator $\widehat{\mu}_B$ of $\mu(\theta)=E[X_1|\theta]$ and the corresponding risk. Hint: A $Beta(a,b)$-distributed random variable θ satisfies the relations $E\theta=\frac{a}{a+b}$ and $Var\ \theta=\frac{ab}{(a+b+1)(a+b)^2}$.

Exercise 4

Consider the ith policy in a heterogeneity model. We suppress the dependence on i in the notation. We assume the heterogeneity parameter θ to be $N(\mu,\sigma^2)$-distributed. Given θ, the claim sizes $X_1,\dots,X_n$ are iid log-normal(θ,τ)-distributed. This means that $\log X_t$ has a representation $\log X_t=\theta+\tau Z_t$ for an iid $N(0,1)$ sequence $\{X_t\}_{t=1}^n$ independent of θ and some positive constant τ.

(a) Calculate the conditional density $f_\theta(y|\vec{X}=\vec{x})$ of θ given

$$\vec{X}=(X_1,\dots,X_n)'=\vec{x}=(x_1,\dots,x_n)'.$$

(b) Calculate the Bayes estimator $\widehat{\mu}_B$ of $\mu(\theta)=E[X_1|\theta]$ and the corresponding risk. It is useful to remember that

$$Ee^{a+bZ_1}=e^{a+b^2/2}\text{ and }Var\ e^{a+bZ_1}=e^{2a+b^2}(e^{b^2}-1),\quad a\in\mathbb{R},b>0.$$