STK4540: Non-life Insurance Mathematics

Extra exercise list

Exercise 1

Consider the total claim amount $S(t) = \sum_{i=1}^{N(t)} X_i$ and assume $S(t)$ has finite exponential moments. The so-called Esscher premium calculation principle is an insurance premium principle defined as

$$p_{Ess}(t) = \frac{E\left[S(t)e^{\delta S(t)}\right]}{E\left[e^{\delta S(t)}\right]}, \quad \delta > 0.$$

(a) Prove that this premium is higher than the net premium, i.e.

$$\frac{E\left[S(t)e^{\delta S(t)}\right]}{E\left[e^{\delta S(t)}\right]} \geqslant E[S(t)], \quad \delta > 0.$$

(b) Prove that in the Cramér-Lundberg model for the total claim amount $S(t)$ we have

$$p_{Ess}(t) = \lambda E[X_1 e^{\delta X_1}]t.$$

(c) Consider the Cramér-Lundberg model again. Show that the Net Profit Condition is satisfied with premium rate $c = \lambda E[X_1 e^{\delta X_1}]$ obtained by the Esscher principle.

Exercise 2

In the renewal model we have the following CLT for $S(t)$ if $Var[X_1] < \infty$ and $Var[W_1] < \infty$,

$$\sup_{x \in \mathbb{R}} \left| P\left(\frac{S(t) - E[S(t)]}{\sqrt{Var[S(t)]}} \leqslant x \right) - \Phi(x) \right|$$
$$= \sup_{y \in \mathbb{R}} \left| P\left(S(t) \leqslant y\right) - \Phi\left(\frac{y - E[S(t)]}{\sqrt{Var[S(t)]}} \leqslant x \right) \right| \to 0,$$

which tells us that our portfolio will eventually converge to a normal distribution in law. That is

$$P(S(t) \leqslant y) \approx \Phi\left(\frac{y - E[S(t)]}{\sqrt{Var[S(t)]}} \leqslant x \right)$$

This serves to construct approximate confidence intervals and hypothesis tests but it does *not* tell us a thing about the error we encounter in the above convergence result. The following is a classical result of the rate and error of convergence of the CLT under slightly stronger conditions:

Theorem (Berry-Esseen inequality). *Let $X_1, \dots, X_n$ be a sequence of i.i.d. random variables with $E[X_1] < \infty$, $Var[X_1] < \infty$ and $E[|X_1|^3] < \infty$. Define $S_0 = 0$, $S_n = X_1 + \cdots + X_n$, $n \geqslant 1$. Then*

$$\sup_{x\in\mathbb{R}} \left| P\left(\frac{\frac{1}{n}S_n - E[X_1]}{\sqrt{Var[X_1]/n}} \leqslant x \right) - \Phi(x) \right| \leqslant \frac{c}{\sqrt{n}} \frac{E[|X_1 - E[X_1]|^3]}{(Var[X_1])^3},$$

where c is a universal constant that you can take as $c = 0.4748$ as per 2012 and Φ is the cumulative distribution function of a standard normal random variable.

You may think this is just the central limit theorem, which it is, but the importance of this result is that it provides an exact error of convergence which, is uniform in x! Therefore its high importance. Actually, this result has been improved over the years and several mathematicians have found sharper bounds, here is a summary of the state of the inequality which looks like a race:

Constant c	Year (author)
7.59	1942 (Essen)
0.7882	1972 (van Beek)
0.7655	1986 (Shiganov)
0.7056	2007 (Shevtsova)
0.7005	2008 (Shevtsova)
0.5894	2009 (Tyurin)
0.5129	2010 (Korolev & Shevtsova)
0.4785	2010 (Tyurin)
0.4748	2012 (Korolev & Shevtsova)

Esseen also proved in 1956 that $c \geqslant 0.40973$, so this race is about to end :)

Nevertheless, we can only apply the Berry-Essen bound in the case of $S(t)|N(t) = n(t)$ conditional on a known number of claims $N(t) = n(t)$ which does not solve the original problem for the unconditional distribution of $S(t)$. In such case we would have

$$\sup_{x\in\mathbb{R}} \left| P\left(\frac{\frac{1}{n(t)}S(t) - E[X_1]}{\sqrt{Var[X_1]/n(t)}} \leqslant x \right) - \Phi(x) \right| \leqslant \frac{c}{\sqrt{n(t)}} \frac{E[|X_1 - E[X_1]|^3]}{(Var[X_1])^3}.$$

From this, we can see that the approximation is very good around the mean $E[S(t)]$ but it shows how dangerous it is to use the CLT when it comes to considering probabilities

$$P(S(t) > y|N(t) = n(t)) = P\left(\frac{\frac{1}{n(t)}S(t) - E[X_1]}{\sqrt{Var[X_1]/n(t)}} > \frac{\frac{1}{n(t)}y - E[X_1]}{\sqrt{Var[X_1]/n(t)}} \right)$$

for large y. The normal approximation is poor if $x = \frac{\frac{1}{n(t)}y - E[X_1]}{\sqrt{Var[X_1]/n(t)}}$ is too large.

Consider an i.i.d. sample $X_1, \dots, X_n$ and the corresponding empirical distribution function:

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\{X_i \leqslant x\}}.$$

By X^* we denote any random variable with distribution function F_n, given $X_1, \dots, X_n$.

(a) Calculate the expectation, the variance and the third absolute moment of X^*.

(b) For (conditionally) i.i.d. random variables X_i^*, $i = 1, \dots, n$ with distribution function F_n calculate the mean and variance of the sample $\overline{X}_n^* = n^{-1} \sum_{i=1}^n X_i^*$.

(c) Apply the strong law of large numbers to show that the limits of $E^*[\overline{X}_n^*]$ and $nVar^*[\overline{X}_n^*]$ as $n \to \infty$ exist and coincide with their deterministic counterparts $E[X_1]$ and $Var[X_1]$, provided the latter quantities are finite. Here, E^* and Var^* refer to expectation and variance with respect to the distribution function F_n of the (conditionally) i.i.d. random variables X_i^*'s.

(d) Apply the Berry-Esseen inequality to

$$\begin{aligned} P^* &\left(\frac{\sqrt{n}}{\sqrt{Var^*[X_1^*]}} (\overline{X}_n^* - E^*[\overline{X}_n^*]) \leqslant x \right) - \Phi(x) \\ &= P \left(\frac{\sqrt{n}}{\sqrt{Var^*[X_1^*]}} (\overline{X}_n^* - E^*[\overline{X}_n^*]) \leqslant x \,\middle|\, X_1, \dots, X_n \right) - \Phi(x), \end{aligned}$$

and show that the (conditional) central limit theorem applies to (X_i^*) if $E[|X_1|^3] < \infty$, i.e., the above differences converge to 0 with probability one.

Exercise 3

Given independent claim sizes $X_1, \dots, X_n$, $n \geqslant 1$ with common distribution function F it is very relevant for the insurer to know the distribution of the maximum of the claims, i.e. $M_n = \max\{X_1, \dots, X_n\}$. A distribution function F is said to be an *extreme value distribution* if it satisfies the following property: for every $n \geqslant 1$ there exist constants $c_n > 0$ and $d_n \in \mathbb{R}$ such that for i.i.d. random variables X_i with common distribution F,

$$\frac{M_n - d_n}{c_n} \overset{\text{law}}{=} X_1.$$

(a) Verify that the Gumbel distribution with distribution function $\Lambda(x) = e^{-e^{-x}}$, $x \in \mathbb{R}$, the Fréchet distribution with distribution function $\Phi_\alpha(x) = e^{-x^{-\alpha}}$, $x > 0$ for some $\alpha > 0$ and the Weibull distribution with distribution function $\Psi_\alpha(x) = e^{-|x|^\alpha}$, $x < 0$ for some $\alpha > 0$ are extreme value distributions. It can be shown that, upto changes of scale and location, these three distributions are the only extreme value distributions.

(b) The extreme value distributions are known to be the only non-degenerate limit distributions for partial maxima $M_n = \max\{X_1, \dots, X_n\}$ of i.i.d. random variables X_i after a suitable scaling and centering, i.e., there exist $c_n > 0$ and $d_n \in \mathbb{R}$ such that

$$\frac{M_n - d_n}{c_n} \overset{\text{law}}{\to} Y \sim H \in \{\Lambda, \Phi_\alpha, \Psi_\alpha\}.$$

This result is known as the Fisher–Tippett–Gnedenko theorem and it is a fundamental result in extreme value theory.

Find suitable constants $c_n > 0$, $d_n \in \mathbb{R}$ and extreme value distributions H such that the above convergence holds for the Pareto, exponentially distributed, and uniformly distributed claim sizes.

Exercise 4

Consider the total claim amount

$$S = \sum_{i=1}^{N} X_i,$$

where $N \sim$ Bin (M, p) for some integer $M \geqslant 0$ and probability p and the claim sizes $X_i \sim$ Ber (q) for some probability q.

(a) Let $q_n = P(N = n)$, $n \geqslant 0$. Show that

$$q_n = \left(a + \frac{b}{n}\right) q_{n-1}, \quad n \geqslant 1,$$

with $a = -\frac{p}{1-p}$ and $b = \frac{p(M+1)}{1-p}$.

(b) By the previous fact and since S is discrete (X_i's are Bernoulli) we can use Panjer's recursion scheme to find the exact probabilities $p_n = P(S = n)$. Show that

$$\begin{cases} p_0 = (1 - pq)^M, \\ p_n = \frac{pq}{1-pq}\left(\frac{M+1}{n} - 1\right) p_{n-1}, \quad n \geqslant 1. \end{cases}$$

(c) Show that $S \sim$ Bin (M, pq). (You can do this using item (*b*) or directly). Compute $E[S]$ and $Var[S]$.

(d) It is well-known that if $Mpq \geqslant 10$ and $M(1 - pq) \geqslant 10$ then by the CLT we have a fairly good approximation with the normal distribution, i.e.

$$\frac{S - E[S]}{\sqrt{Var[S]}} \approx N(0, 1).$$

Assume $M = 100$, $p = 0.5$ and $q = 0.2$. Use this to compute the probabilities $P(S \geqslant 10)$, $P(S \geqslant 20)$ and $P(S \geqslant 30)$ using the exact value and the normal approximation. What do you observe?

Exercise 5

Consider an insurance policy with claims $X_i \sim \text{U}[0,\theta]$ where θ is unknown and we suppose θ is Pareto distributed with location parameter $x_m > 0$ and scale parameter α. The density functions, expectations and variances are given by

$$f_{X_i|\theta}(x|\theta) = \frac{1}{\theta}, \quad 0 \leqslant x \leqslant \theta, \quad E[X_i|\theta] = \frac{\theta}{2}, \quad Var[X_i|\theta] = \frac{\theta^2}{12},$$
$$f_\theta(y) = \alpha\frac{x_m^\alpha}{y^{\alpha+1}}, \quad y \geqslant x_m, \quad E[\theta] = \frac{\alpha x_m}{\alpha - 1}, \quad Var[\theta] = \frac{\alpha x_m^2}{(\alpha-1)^2(\alpha-2)},$$

where the Pareto variable has finite expectation only if $\alpha > 1$ and finite variance only if $\alpha > 2$.

Observe that θ provides a (random) bound for the largest claim. A frequestist approach would be to take θ as the maximum of a sample. Here, instead we add some uncertainty and update the distribution of θ after observations. Do not think that claims are uniformly distributed or bounded variables, they are actually not!

We observe n claims of sizes $X_1, \ldots, X_n$. Denote $X = (X_1, \ldots, X_n)$ and similarly $x = (x_1, \ldots, x_n)$ to relax notation.

(a) In the Bayesian setting it is in general not easy to find the unconditional distribution for the claim sizes, i.e. we have $X_1|\theta$ and we have θ but we do not know X. In general we do not need to know it. Nevertheless, find the (unconditional) distribution of X_1 and comment.

(b) Show that the posterior distribution, i.e. $\theta|X_1, \ldots, X_n$ is given by

$$\theta|X \sim \text{Pareto } (x_m^*, \alpha^*)$$

where

$$x_m^* = \max\{X_1, \ldots, X_n, x_m\}, \quad \alpha^* = \alpha + n.$$

(c) Find the Bayes estimator, denoted by $\widehat{\mu}_B$, for the net premium $\mu(\theta) = E[X_1|\theta]$ and show that $\widehat{\mu}_B$ converges to $\mu(\theta)$ almost surely. Compute the Bayes risk too.

(d) Assume a prior on θ Pareto distributed with $x_m = 10$ and $\alpha = 4$. We observe $x_1 = 2, x_2 = 5, x_3 = 12, x_4 = 4, x_5 = 5, x_6 = 2, x_7 = 8, x_8 = 5, x_9 = 5, x_{10} = 8$. Update your prior belief and compute the Bayes estimator based on this sample.

(e) Under the same setting as in (c) now imagine you have a portfolio $S(t) = \sum_{i=1}^{N(t)} X_i$ with Poisson claims $N(t) \sim \text{Pois }(\lambda t)$, and X_i are also uniformly distributed with parameter θ and θ is Pareto as in (c). Considering the ruin process $U(t) = u + ct - S(t)$ with initial capital $u > 0$ and premium rate $c > 0$. Find the premium you should charge in order to avoid ruin with probability one if no data is available, and if data has been collected. What do you observe?