STK4540: Non-life Insurance Mathematics

Final exercise list

Exercise 1

Consider an insurance policy with total claim amount process given by

$$S(t) = \sum_{i=1}^{N(t)} X_i,$$

where t is the time measured in, say years, N is a mixed Poisson process with mixing variable $\theta \sim \Gamma(\gamma, \beta)$ and $\mu(t) = t$ (see p. 66) and X_i are i.i.d. with

$$P(X_i = 0) = 0.1, \quad P(X_i = 1) = 0.6, \quad P(X_i = 2) = 0.3.$$

We are interested in computing the probabilities $p_n(t) := P(S(t) = n)$ for each $n \geqslant 0$. We will use a Panjer recursion scheme.

(a) Prove that

$$p_0(t) = \left(\frac{p}{1 + (1-p)P(X_1 = 0)}\right)^v,$$

where $p = \frac{\beta}{t+\beta}$ and $v = \gamma$.

(b) Compute $p_1(t)$ and $p_2(t)$ in order to find a general expression for $p_n(t)$, $n \geqslant 3$.

(c) Compute the probability that no X_i is 1 during the first year.

(d) For $t = 1$, $p = 0.5$ and $v = 3$, find a normal approximation for the distribution of $S(1)$. Compute $P(S(1) \leqslant 2)$ using the exact method and your approximation and compare.

Exercise 2

Before starting with the exercise we may need some definitions and formulas, presented here under:

A random variable Y is said to have a *Gompertz distribution* if it is absolutely continuous and has a density function given by

$$f_Y(y) = b\eta e^{\eta} e^{by} e^{-\eta e^{by}}, \quad y \in [0, \infty).$$

The parameter $b > 0$ is called the scale parameter and η is the shape parameter. The mean of this random variable is given by

$$E[Y] = \frac{1}{b} e^{\eta} \int_{\eta}^{\infty} \frac{e^{-v}}{v} dv.$$

We can see that this is a very complex expression which can not be calculated in closed form.

A random variable Z is said to have a *gamma distribution* if it is absolutely continuous and has a density function given by

$$f_Z(z) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} z^{\alpha-1} e^{-\beta z}, \quad z \in [0, \infty).$$

The parameter $\alpha > 0$ is called the shape parameter and β is the scale parameter. For Z we have

$$E[Z] = \frac{\alpha}{\beta}, \quad Var[Z] = \frac{\alpha}{\beta^2}.$$

Consider the heterogeneity model for an insurance company with several policies. We will focus on one given policy. Assume that we have some past history of this policy given by a sequence $X_1, \ldots, X_n$, $n \geqslant 1$ of Gompertz distributed random variables with scale parameter b and shape parameter η where η is the heterogeneity parameter following a gamma distribution with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$.

(a) Show that the random variable $\eta | \vec{X} = \vec{x}$ with $\vec{X} = (X_1, \ldots, X_n)'$ and $\vec{x} = (x_1, \ldots, x_n)' \in \mathbb{R}^n$ is gamma distributed with parameters

$$\overline{\alpha} = n + \alpha \quad \overline{\beta} = \beta - n + \sum_{i=1}^{n} e^{bx_i}.$$

(c) Denote the net premium $\mu(\eta) = E[X_1|\eta]$. Prove that the Bayes estimator of $\mu(\eta)$ is given by

$$\widehat{\mu}_B = \left(\frac{\overline{\beta}}{\overline{\beta}-1}\right)^{\overline{\alpha}} \frac{1}{b\Gamma(\overline{\alpha})} \int_0^{\infty} \gamma(\overline{\alpha}, (\overline{\beta}-1)v) \frac{e^{-v}}{v} dv,$$

where $\gamma(s, x) = \int_0^x t^{s-1} e^{-t} dt$ is the *incomplete gamma function.*

The Bayes estimator in this case is given by a quite complicated expression. In practice, this is not an issue, since we can easily evaluate the integrals using a computer.

Exercise 3

Consider the heterogeneity model for an insurance company with several policies. We will focus on one given policy. Assume that we have some past history of this policy given by a sequence $X_1, \ldots, X_n$, $n \geqslant 1$ of exponentially distributed random variables with parameter θ where θ is the heterogeneity parameter following a gamma distribution with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$.

(a) Find the net premium for this policy based on the past observations $X_1, \ldots, X_n$ and its risk.

(b) Find the best approximation for the net premium on the class of random variables

$$\mathcal{L} = \left\{ Y = a_0 + \sum_{i=1}^{n} a_i X_i \right\}.$$

What do you observe compared to the previous exercise?

(c) Find the best approximation for the net premium in the class

$$\mathcal{Q} = \left\{ Y = a_0 + \sum_{i=1}^{n} a_i X_i^2 \right\}.$$